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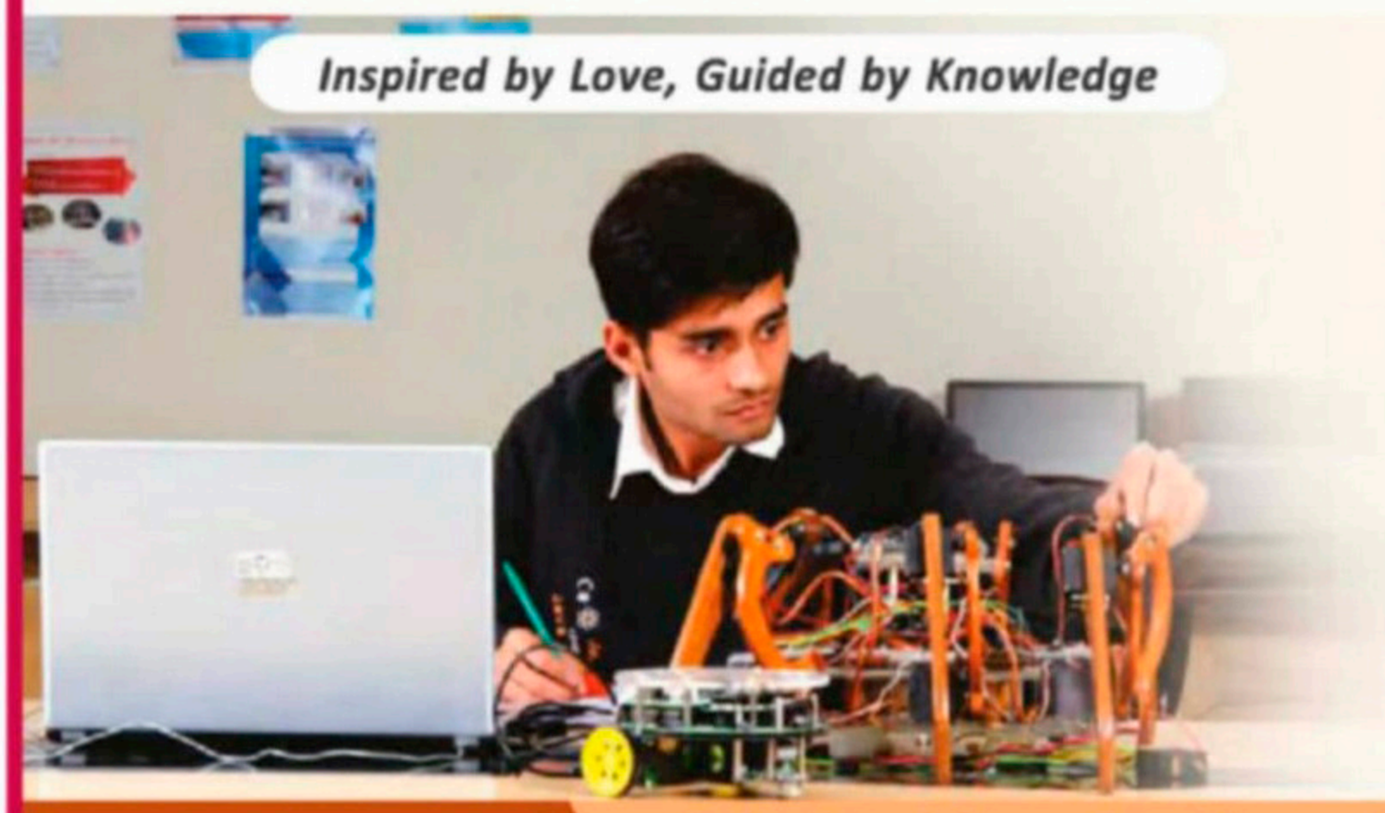

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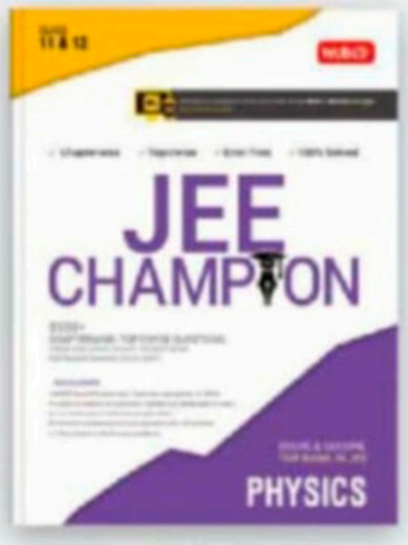


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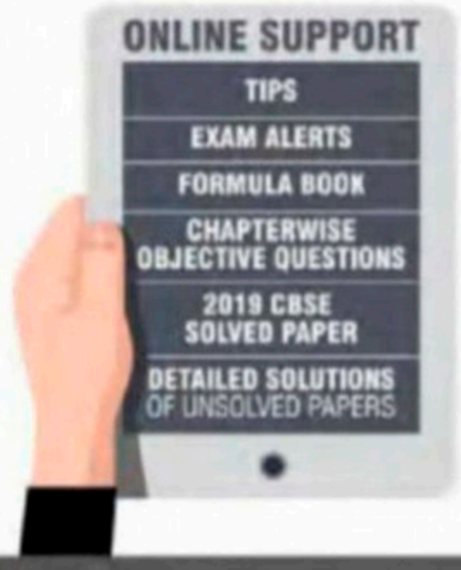


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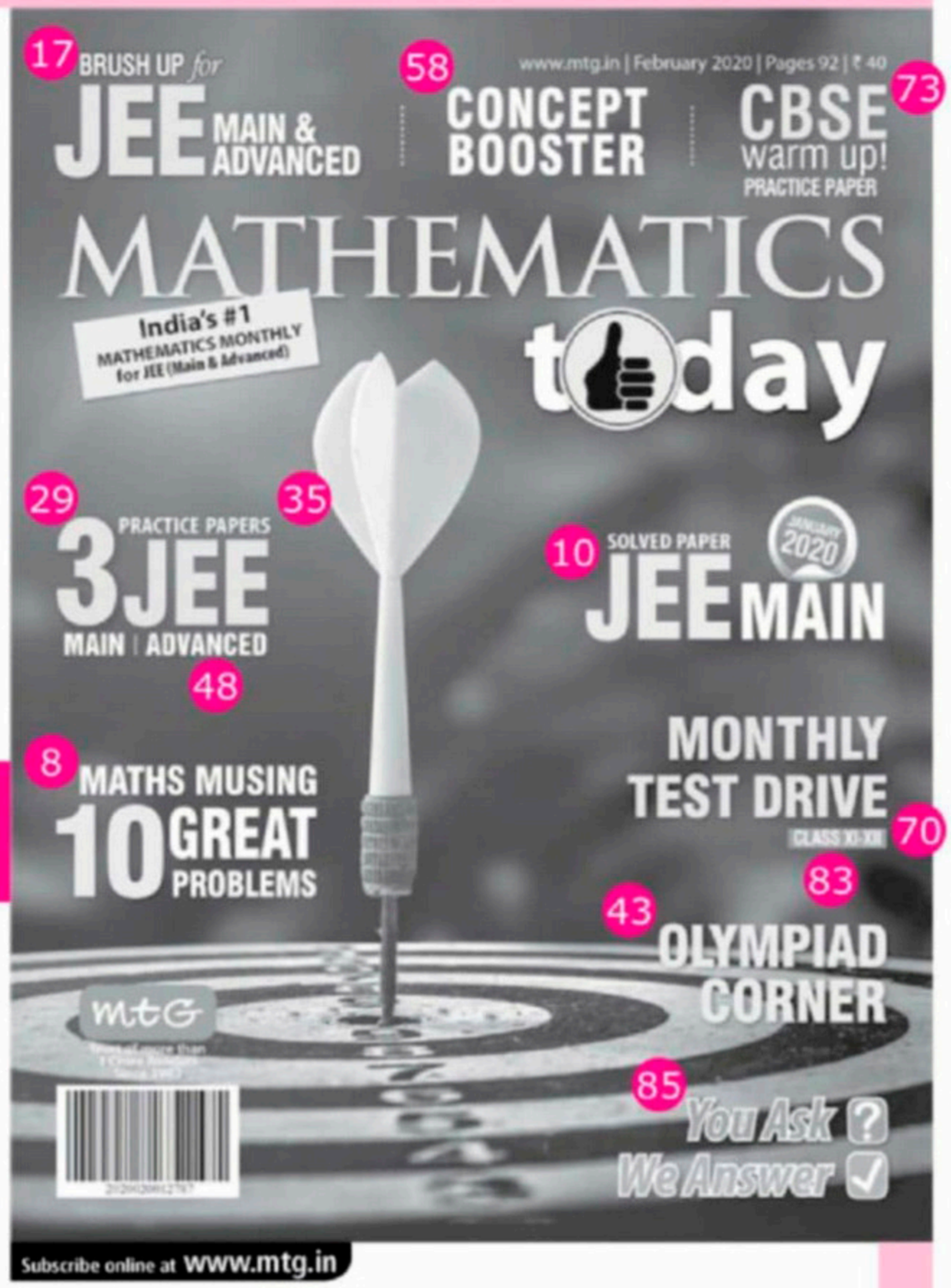


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*During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.*

*Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.*

## PROBLEM Set 206

### JEE MAIN

**1.** If $a, b, c$ are in H.P., $b, c, d$ are in G.P. and $c, d, e$ are in A.P., then $\frac{ab^2}{(2a-b)^2}$ is equal to

(a) $b$ (b) $a$ (c) $e$ (d) $d$

**2.** If $z$ and $w$ are two non-zero complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then $\bar{z}w$ is equal to

(a) 1 (b) $-1$ (c) $i$ (d) $-i$

**3.** In the expansion of $(7^{1/3} + 11^{1/9})^{6561}$, the number of terms free from radical is

(a) 715 (b) 725 (c) 730 (d) 745

**4.** Five digit numbers are formed using the digits 0, 1, 2, 3, 4, 6, 7 without repetition. Find the probability that a number is divisible by 3.

(a) 5/18 (b) 3/17 (c) 4/13 (d) 2/19

**5.** Let $f(x) = \int_0^x \frac{dt}{\sqrt{1+t^3}}$ and $g(x)$ be the inverse function of $f(x)$, then $g'' = \lambda g^2$. The value of $\lambda =$

(a) 1/2 (b) 3/2 (c) 2/3 (d) 1

### JEE ADVANCED

**6.** Let $\vec{x}, \vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\pi/3$. If $\vec{a}$ is a non-zero vector perpendicular to $\vec{x}$ and $\vec{y} \times \vec{z}$ and $\vec{b}$ is non-zero vector perpendicular to $\vec{y}$ and $\vec{z} \times \vec{x}$, then

(a) $\vec{b} = (\vec{b} \cdot \vec{z})(\vec{z} - \vec{x})$ (b) $\vec{a} = (\vec{a} \cdot \vec{y})(\vec{y} - \vec{z})$

(c) $\vec{a} \cdot \vec{b} = -(\vec{a} \cdot \vec{y})(\vec{b} \cdot \vec{z})$ (d) $\vec{a} = (\vec{a} \cdot \vec{y})(\vec{z} - \vec{y})$

### COMPREHENSION

Two lines whose equations are $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{\lambda}$ and $\frac{x-2}{3} = \frac{y-3}{2} = \frac{z-2}{3}$ lie in the same plane.

**7.** The value of $\sin^{-1}(\sin \lambda)$ is equal to

(a) 3 (b) $\pi - 3$ (c) 4 (d) $\pi - 4$

**8.** Point of intersection of the lines lies on

(a) $3x + y + z = 20$ (b) $2x + y + z = 25$

(c) $3x + 2y + z = 24$ (d) $x = y = z$

### NUMERICAL VALUE TYPE

**9.** A person purchases one kilogram of tomatoes from each of the 4 places at the rate of 1 kg, 2 kg, 3 kg and 4 kg per rupee respectively. On the average he has purchased $x$ kg of tomatoes per rupee, then the value of $x$ is

### MATRIX MATCH

**10.** $z_1, z_2, z_3$ are the vertices of a triangle.

Match the triangles given in column-II with the conditions given in column-I:

| Column-I | | Column-II | |
|---|---|---|---|
| P. | $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_1z_3$ | 1. | Right angled |
| Q. | $\text{Re}\frac{z_3 - z_1}{z_3 - z_2} = 0$ | 2. | Obtuse angled |
| R. | $\text{Re}\frac{z_3 - z_1}{z_3 - z_2} < 0$ | 3. | Isosceles and right angled |
| S. | $\frac{z_3 - z_1}{z_3 - z_2} = i$ | 4. | Equilateral |

| | P | Q | R | S |
|---|---|---|---|---|
| (a) | 1 | 2 | 3 | 4 |
| (b) | 2 | 3 | 4 | 1 |
| (c) | 4 | 1 | 2 | 3 |
| (d) | 3 | 4 | 1 | 2 |

See Solution Set of Maths Musing 205 on page no. 57

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### Knowledge Quiz - 27

Find the number of natural numbers which are smaller than $2\times10^8$ and which can be written by means of the digits 1 and 2.

**Please send your detailed solution before 12th February 2020 to quiz@kcseducate.in along with your name, father's name, class, school, address and contact number.**

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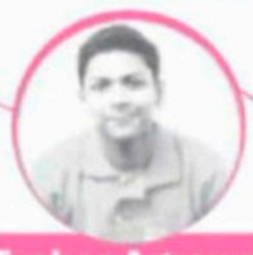

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# SOLVED PAPER

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# JEE MAIN 2020

1. The area of the region, enclosed by the circle $x^2 + y^2 = 2$ which is not common to the region bounded by the parabola $y^2 = x$ and the straight line $y = x$, is

(a) $\frac{1}{3}(12\pi - 1)$ (b) $\frac{1}{6}(12\pi - 1)$

(c) $\frac{1}{3}(6\pi - 1)$ (d) $\frac{1}{6}(24\pi - 1)$

2. Total number of 6-digit numbers in which only and all the five digits 1, 3, 5, 7 and 9 appear, is

(a) $5^6$ (b) $\frac{1}{2}(6!)$ (c) $6!$ (d) $\frac{5}{2}(6!)$

3. An unbiased coin is tossed 5 times. Suppose that a variable $X$ is assigned the value $k$ when $k$ consecutive heads are obtained for $k = 3, 4, 5$, otherwise $X$ takes the value $-1$. Then the expected value of $X$, is

(a) 1/8 (b) 3/16 (c) −1/8 (d) −3/16

4. If $\text{Re}\left(\frac{z-1}{2z+i}\right)=1$, where $z = x + iy$, then the point $(x, y)$ lies on a

(a) circle whose centre is at $\left(-\frac{1}{2},-\frac{3}{2}\right)$.

(b) straight line whose slope is $\frac{3}{2}$.

(c) circle whose diameter is $\frac{\sqrt{5}}{2}$.

(d) straight line whose slope is $-\frac{2}{3}$.

5. If $f(a + b + 1 - x) = f(x)$, for all $x$, where $a$ and $b$ are fixed positive real numbers, then $\frac{1}{a+b}\int_a^b x(f(x) + f(x+1))dx$ is equal to

(a) $\int_{a-1}^{b-1} f(x)dx$ (b) $\int_{a+1}^{b+1} f(x+1)dx$

(c) $\int_{a-1}^{b-1} f(x+1)dx$ (d) $\int_{a+1}^{b+1} f(x)dx$

6. If the distance between the foci of an ellipse is 6 and the distance between its directrices is 12, then the length of its latus rectum is

(a) $2\sqrt{3}$ (b) $\sqrt{3}$ (c) $\frac{3}{\sqrt{2}}$ (d) $3\sqrt{2}$

7. The logical statement $(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$ is equivalent to

(a) $\sim p$ (b) $p$ (c) $q$ (d) $\sim q$

8. The greatest positive integer $k$, for which $49^k + 1$ is a factor of the sum $49^{125} + 49^{124} + .... + 49^2 + 49 + 1$, is

(a) 32 (b) 60 (c) 65 (d) 63

9. A vector $\vec{a} = \alpha\hat{i} + 2\hat{j} + \beta\hat{k}\ (\alpha, \beta \in R)$ lies in the plane of the vectors, $\vec{b} = \hat{i} + \hat{j}$ and $\vec{c} = \hat{i} - \hat{j} + 4\hat{k}$. If $\vec{a}$ bisects the angle between $\vec{b}$ and $\vec{c}$, then

(a) $\vec{a} \cdot \hat{i} + 3 = 0$ (b) $\vec{a} \cdot \hat{k} + 4 = 0$

(c) $\vec{a} \cdot \hat{i} + 1 = 0$ (d) $\vec{a} \cdot \hat{k} + 2 = 0$

10. If $y(\alpha)=\sqrt{2\left(\frac{\tan\alpha+\cot\alpha}{1+\tan^2\alpha}\right)+\frac{1}{\sin^2\alpha}}, \alpha\in\left(\frac{3\pi}{4}, \pi\right)$, then $\frac{dy}{d\alpha}$ at $\alpha=\frac{5\pi}{6}$ is

(a) $-\frac{1}{4}$ (b) $\frac{4}{3}$ (c) 4 (d) −4

11. If $y = mx + 4$ is a tangent to both the parabolas, $y^2 = 4x$ and $x^2 = 2by$, then $b$ is equal to

(a) −64 (b) 128 (c) −128 (d) −32

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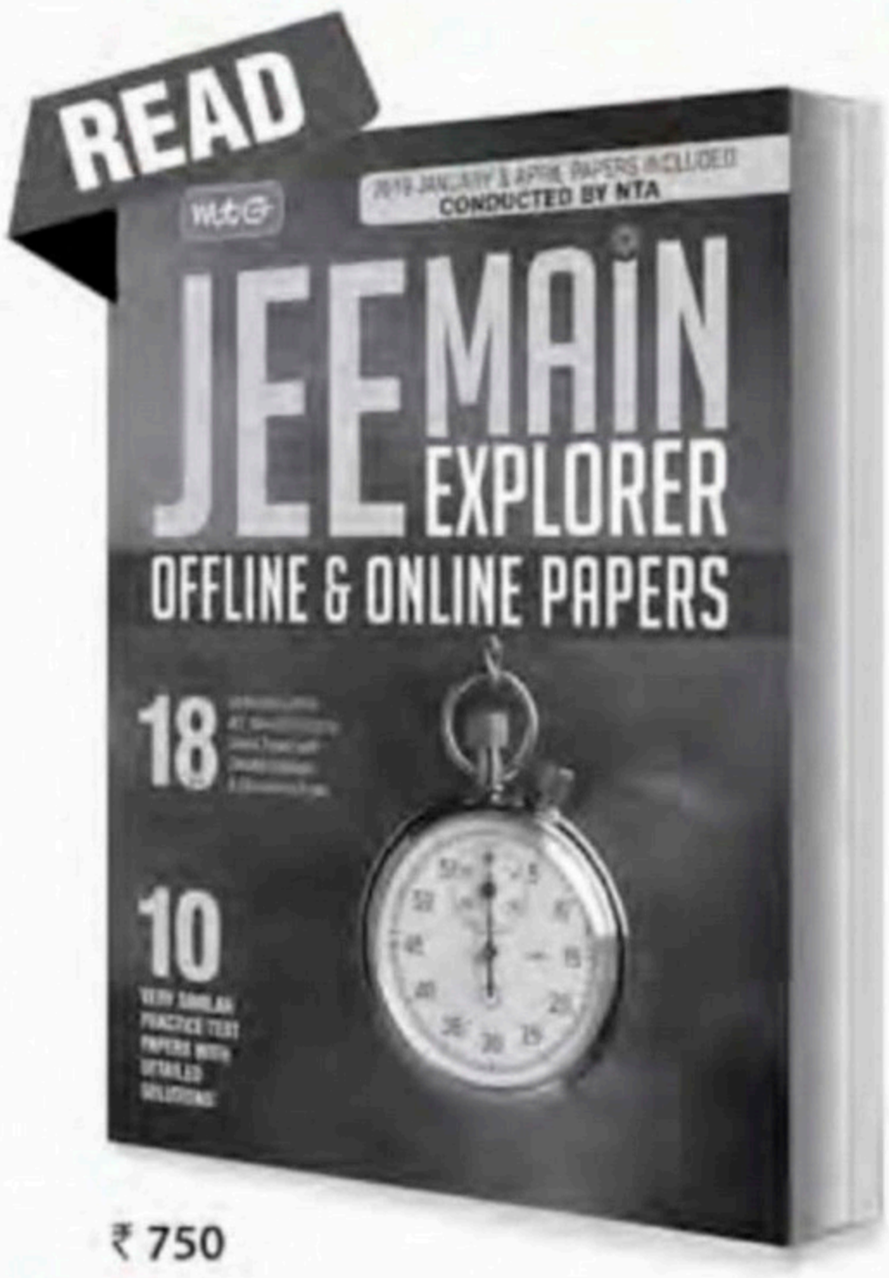


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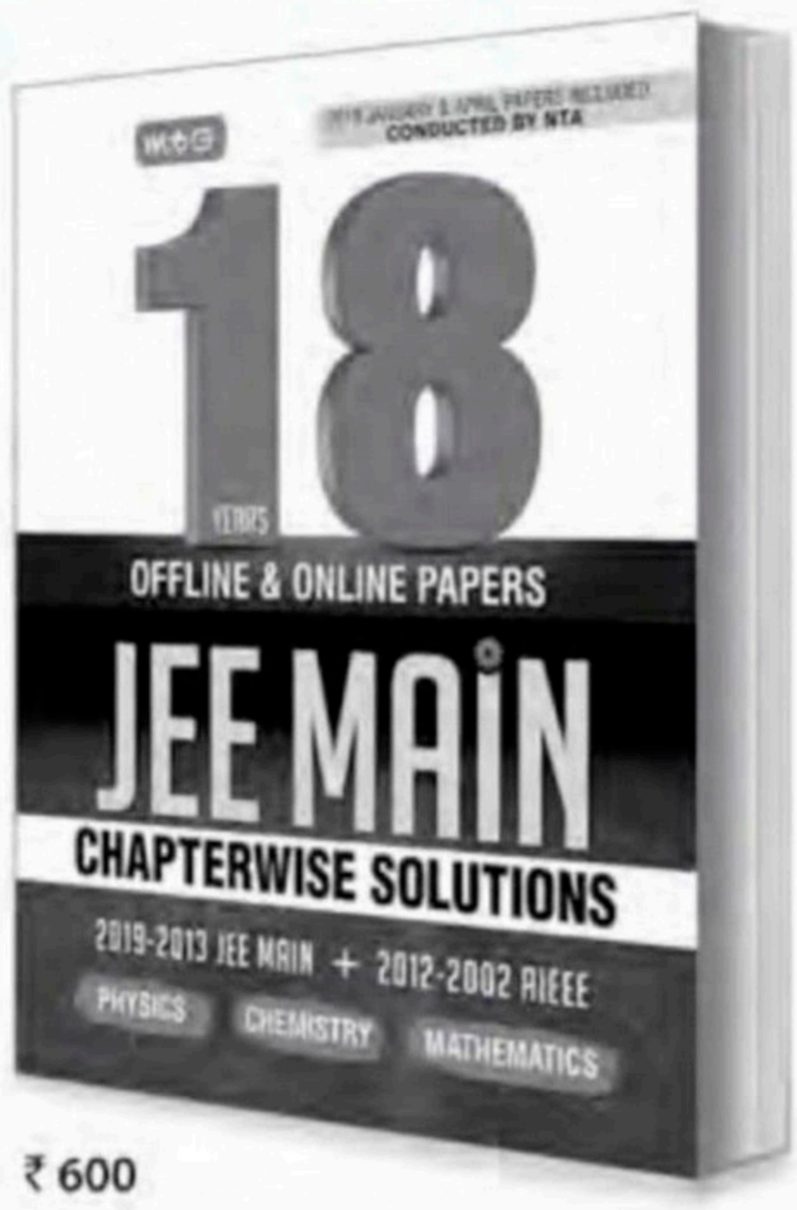


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**12.** Let $\alpha$ be a root of the equation $x^2 + x + 1 = 0$ and the matrix $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha^4 \end{bmatrix}$, then the matrix $A^{31}$ is equal to

(a) $A$ (b) $A^2$ (c) $A^3$ (d) $I_3$

**13.** If $g(x) = x^2 + x - 1$ and $(gof)(x) = 4x^2 - 10x + 5$, then $f\left(\frac{5}{4}\right)$ is equal to

(a) $-\frac{3}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) $\frac{3}{2}$

**14.** Let $\alpha$ and $\beta$ be two real roots of the equation $(k+1)\tan^2 x - \sqrt{2}\cdot\lambda\tan x = (1-k)$, where $k(\neq -1)$ and $\lambda$ are real numbers. If $\tan^2(\alpha+\beta) = 50$, then a value of $\lambda$ is

(a) $5\sqrt{2}$ (b) $10\sqrt{2}$ (c) 10 (d) 5

**15.** Let $P$ be a plane passing through the points (2, 1, 0), (4, 1, 1) and (5, 0, 1) and $R$ be any point (2, 1, 6). Then the image of $R$ in the plane $P$ is

(a) (6, 5, 2) (b) (6, 5, –2)
(c) (4, 3, 2) (d) (3, 4, –2)

**16.** Let $x^k + y^k = a^k$, $(a, k > 0)$ and $\frac{dy}{dx} + \left(\frac{y}{x}\right)^{\frac{1}{3}} = 0$, then $k$ is

(a) $\frac{1}{3}$ (b) $\frac{3}{2}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

**17.** Let the function, $f : [-7, 0] \to R$ be continuous on $[-7, 0]$ and differentiable on $(-7, 0)$. If $f(-7) = -3$ and $f'(x) \le 2$, for all $x \in (-7, 0)$, then for all such functions $f$, $f(-1) + f(0)$ lies in the interval

(a) $[-6, 20]$ (b) $(-\infty, 20]$
(c) $(-\infty, 11]$ (d) $[-3, 11]$

**18.** If $y = y(x)$ is the solution of the differential equation, $e^y\left(\frac{dy}{dx} - 1\right) = e^x$ such that $y(0) = 0$, then $y(1)$ is equal to

(a) $\log_e 2$ (b) $2e$
(c) $2 + \log_e 2$ (d) $1 + \log_e 2$

**19.** Five numbers are in A.P., whose sum is 25 and product is 2520. If one of these five numbers is $-\frac{1}{2}$, then the greatest number amongst them is

(a) 16 (b) 27 (c) 7 (d) $\frac{21}{2}$

**20.** If the system of linear equations
$2x + 2ay + az = 0$
$2x + 3by + bz = 0$
$2x + 4cy + cz = 0$, where $a, b, c \in R$ are non-zero and distinct; has a non-zero solution, then

(a) $a + b + c = 0$ (b) $a, b, c$ are in A.P.
(c) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. (d) $a, b, c$ are in G.P.

**21.** $\lim\limits_{x\to 2} \frac{3^x + 3^{3-x} - 12}{3^{-x/2} - 3^{1-x}}$ is equal to _____.

**22.** If the variance of the first $n$ natural numbers is 10 and the variance of the first $m$ even natural numbers is 16, then $m + n$ is equal to _____.

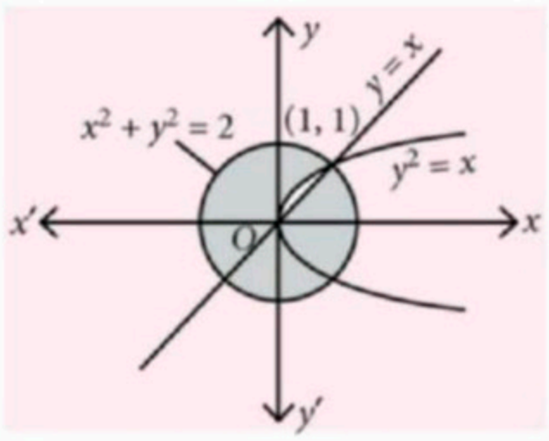
**23.** If the sum of the coefficients of all even powers of $x$ in the product $(1 + x + x^2 + .... + x^{2n})(1 - x + x^2 - x^3 + ... + x^{2n})$ is 61, then $n$ is equal to _____.

**24.** Let $S$ be the set of points where the function, $f(x) = |2 - |x - 3||$, $x \in R$, is not differentiable. Then $\sum\limits_{x\in S} f(f(x))$ is equal to _____.

**25.** Let $A(1, 0)$, $B(6, 2)$ and $C\left(\frac{3}{2}, 6\right)$ be the vertices of a triangle $ABC$. If $P$ is a point inside the triangle $ABC$ such that the triangles $APC$, $APB$ and $BPC$ have equal areas, then the length of the line segemnt $PQ$, where $Q$ is the point $\left(-\frac{7}{6}, -\frac{1}{3}\right)$, is _____.

## SOLUTIONS

**1. (b) :** Required area = Total area of circle – area enclosed between $y = x$ and $y^2 = x$




$$= 2\pi - \int_0^1 (\sqrt{x} - x)\,dx$$

$$= 2\pi - \left[\frac{2x^{3/2}}{3} - \frac{x^2}{2}\right]_0^1$$

$$= 2\pi - \left(\frac{2}{3} - \frac{1}{2}\right) = 2\pi - \left(\frac{1}{6}\right) = \frac{12\pi - 1}{6}$$

**2. (d) :** Given digits are 1, 3, 5, 7, 9
For digit to repeat we have ${}^5C_1$ ways and six digits can be arranged in $\frac{6!}{2!}$ ways.

Hence, total six digit numbers = $\frac{5(6!)}{2!}$

**3. (a) :** Total number of possible outcomes $= 2^5 = 32$

Probability distribution for $X$ is as follows:

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $P(k)$ | $\frac{1}{32}$ | $\frac{12}{32}$ | $\frac{11}{32}$ | $\frac{5}{32}$ | $\frac{2}{32}$ | $\frac{1}{32}$ |

where $k$ = no. of times head occur consecutively.

Now, $E(X) = \Sigma XP(k) = (-1)\times\frac{1}{32} + (-1)\times\frac{12}{32} + (-1)\times\frac{11}{32} + 3\times\frac{5}{32} + 4\times\frac{2}{32} + 5\times\frac{1}{32} = \frac{1}{8}$

**4. (c) :** Consider $\left(\frac{z-1}{2z+i}\right) = \frac{x+iy-1}{2(x+iy)+i}$

$$= \frac{(x-1)+iy}{2x+(2y+1)i} \times \frac{2x-(2y+1)i}{2x-(2y+1)i}$$

$\therefore$ $\text{Re}\left(\frac{z-1}{2z+i}\right) = \frac{2x(x-1)+y(2y+1)}{(2x)^2+(2y+1)^2} = 1$ [Given]

$\Rightarrow 2x^2 + 2y^2 - 2x + y = 4x^2 + 4y^2 + 4y + 1$

$\Rightarrow 2x^2 + 2y^2 + 2x + 3y + 1 = 0$

$\Rightarrow x^2 + y^2 + x + \frac{3}{2}y + \frac{1}{2} = 0$, which is a circle with centre $\left(\frac{-1}{2}, \frac{-3}{4}\right)$ and radius $(r) = \sqrt{\frac{1}{4}+\frac{9}{16}-\frac{1}{2}} = \frac{\sqrt{5}}{4}$.

**5. (c, d) :** Let $I = \frac{1}{(a+b)}\int_a^b x[f(x)+f(x+1)]dx$ ...(i)

Again,

$$I = \frac{1}{(a+b)}\int_a^b (a+b-x)[f(a+b-x)+f(a+b+1-x)]dx$$

$$\Rightarrow I = \frac{1}{(a+b)}\int_a^b (a+b-x)[f(x+1)+f(x)]dx \quad ...(ii)$$

[Given, $f(a + b + 1 - x) = f(x)$]

Adding (i) and (ii), we get

$$2I = \int_a^b [f(x+1)+f(x)]dx$$

$$\Rightarrow 2I = \int_a^b f(x+1)dx + \int_a^b f(x)dx$$

$$\Rightarrow 2I = \int_a^b f(a+b+1-x)dx + \int_a^b f(x)dx$$

$$\Rightarrow 2I = 2\int_a^b f(x)dx \Rightarrow I = \int_a^b f(x)dx$$

Now, let $x = z + 1 \Rightarrow dx = dz$

When $x = a \Rightarrow z = a - 1$

When $x = b \Rightarrow z = b - 1$

$\therefore$ $I = \int_{a-1}^{b-1} f(z+1)dz$

**6. (d) :** Given, $2ae = 6 \Rightarrow ae = 3$ ...(i)

Again $\frac{2a}{e} = 12 \Rightarrow \frac{a}{e} = 6$ ...(ii)

From (i) and (ii), we have $a^2 = 18$

Now, $b^2 = a^2 - a^2e^2 = 18 - 9 = 9$

$\therefore$ Length of latus rectum $= \frac{2b^2}{a} = \frac{2\times 9}{3\sqrt{2}} = 3\sqrt{2}$

**7. (a) :**

| $p$ | $q$ | $p \Rightarrow q$ | $\sim p$ | $q \Rightarrow \sim p$ | $(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$ |
|---|---|---|---|---|---|
| T | T | T | F | F | F |
| T | F | F | F | T | F |
| F | T | T | T | T | T |
| F | F | T | T | T | T |

Clearly, $(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$ is equivalent to $\sim p$.

**8. (d) :** Let $S = 1 + 49 + 49^2 + ..... + 49^{125}$

$$= \frac{49^{126}-1}{49-1} = \frac{(49^{63}+1)(49^{63}-1)}{48}$$

So, greatest value of $k$ for which $49^k + 1$ is a factor of $S$ is 63.

**9. (d) :** Angle bisector between $\vec{b}$ and $\vec{c}$ can be $\vec{a} = \lambda(\hat{b}+\hat{c})$ or $\vec{a} = \mu(\hat{b}-\hat{c})$

$$\Rightarrow \vec{a} = \lambda\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}} + \frac{\hat{i}-\hat{j}+4\hat{k}}{3\sqrt{2}}\right) = \frac{\lambda}{3\sqrt{2}}\left(3\hat{i}+3\hat{j}+\hat{i}-\hat{j}+4\hat{k}\right)$$

$$= \frac{\lambda}{3\sqrt{2}}\left(4\hat{i}+2\hat{j}+4\hat{k}\right)$$

Comparing with $\vec{a} = \alpha\hat{i} + 2\hat{j} + \beta\hat{k}$, we get

$\frac{2\lambda}{3\sqrt{2}} = 2 \Rightarrow \lambda = 3\sqrt{2}$

$\therefore$ $\vec{a} = 4\hat{i} + 2\hat{j} + 4\hat{k} \Rightarrow \vec{a}\cdot\hat{i} = 4$ and $\vec{a}\cdot\hat{k} = 4$

Which not satisfy any option. So, consider

$$\vec{a} = \mu\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}} - \frac{\hat{i}-\hat{j}+4\hat{k}}{3\sqrt{2}}\right)$$

$$\Rightarrow \vec{a} = \frac{\mu}{3\sqrt{2}}(3\hat{i}+3\hat{j}-\hat{i}+\hat{j}-4\hat{k}) = \frac{\mu}{3\sqrt{2}}(2\hat{i}+4\hat{j}-4\hat{k})$$

Comparing with $\vec{a} = \alpha\hat{i} + 2\hat{j} + \beta\hat{k}$, we get

$\frac{4\mu}{3\sqrt{2}} = 2 \Rightarrow \mu = \frac{3\sqrt{2}}{2}$

$\therefore$ $\vec{a} = \hat{i} + 2\hat{j} - 2\hat{k} \Rightarrow \vec{a}\cdot\hat{i} = 1$

and $\vec{a}\cdot\hat{k} = -2 \Rightarrow \vec{a}\cdot\hat{k} + 2 = 0$

**10. (c) :** Given, $y=\sqrt{\frac{2(\tan\alpha+\cot\alpha)}{1+\tan^2\alpha}+\frac{1}{\sin^2\alpha}}$

$=\sqrt{\frac{2(\tan^2\alpha+1)}{\tan\alpha(1+\tan^2\alpha)}+\frac{1}{\sin^2\alpha}}=\sqrt{2\cot\alpha+\text{cosec}^2\alpha}$

$= |1 + \cot\alpha| = -1 - \cot\alpha$

$\therefore \quad \frac{dy}{d\alpha}=\text{cosec}^2\alpha \Rightarrow \left.\frac{dy}{d\alpha}\right|_{\alpha=\frac{5\pi}{6}}=\text{cosec}^2\left(\frac{5\pi}{6}\right)=4$

**11. (c) :** Given tangent is $y = mx + 4$ ...(i)

Tangent to the parabola $y^2 = 4x$ is $y = mx + \frac{1}{m}$ ...(ii)

From (i) and (ii), we get $4=\frac{1}{m}\Rightarrow m=\frac{1}{4}$

Now, $y=\frac{1}{4}x+4$ is also a tangent to parabola $x^2 = 2by$

$\Rightarrow \quad x^2=2b\left(\frac{x+16}{4}\right)$

$\Rightarrow 2x^2-bx-16b=0$ has equal roots.

$\therefore \quad D = 0 \Rightarrow b^2 - 4 \times 2 \times (-16b) = 0$

$\Rightarrow \quad b^2 + 128b = 0 \Rightarrow b(b + 128) = 0$

$\Rightarrow \quad b = -128$ [$\because$ $b = 0$ is not possible]

**12. (c) :** We have, $\alpha$ is a root of $x^2 + x + 1 = 0$

$\therefore \quad \alpha^2 + \alpha + 1 = 0 \Rightarrow \alpha^3 - 1 = 0 \Rightarrow \alpha^3 = 1$

Given, $A=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \alpha & \alpha^2\\ 1 & \alpha^2 & \alpha^4\end{bmatrix}=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & \alpha & \alpha^2\\ 1 & \alpha^2 & \alpha\end{bmatrix}$

$A^2=\frac{1}{3}\begin{bmatrix}1 & 1 & 1\\ 1 & \alpha & \alpha^2\\ 1 & \alpha^2 & \alpha\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & \alpha & \alpha^2\\ 1 & \alpha^2 & \alpha\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$

$\therefore \quad A^4 = I$

Now, $A^{31} = A^{28} \times A^3 = A^3$

**13. (b) :** Since, $g(x) = x^2 + x - 1$

$\therefore \quad (gof)(x) = g(f(x)) = f^2(x) + f(x) - 1$

$\Rightarrow \quad g\left(f\left(\frac{5}{4}\right)\right)=f^2\left(\frac{5}{4}\right)+f\left(\frac{5}{4}\right)-1$ ...(i)

Again, $g\left(f\left(\frac{5}{4}\right)\right)=4\left(\frac{5}{4}\right)^2-10\left(\frac{5}{4}\right)+5=-\frac{5}{4}$ ...(ii)

From (i) and (ii), we get

$-\frac{5}{4}=f^2\left(\frac{5}{4}\right)+f\left(\frac{5}{4}\right)-1 \Rightarrow f^2\left(\frac{5}{4}\right)+f\left(\frac{5}{4}\right)+\frac{1}{4}=0$

$\Rightarrow \quad \left(f\left(\frac{5}{4}\right)+\frac{1}{2}\right)^2=0 \quad \Rightarrow \quad f\left(\frac{5}{4}\right)=-\frac{1}{2}$

**14. (c) :** Given, $(k + 1)\tan^2 x - \sqrt{2}\lambda\tan x + (k - 1) = 0$

$\therefore \quad \tan\alpha + \tan\beta = \frac{\sqrt{2}\lambda}{k+1}$ and $\tan\alpha\cdot\tan\beta=\frac{k-1}{k+1}$

Now, $\tan(\alpha + \beta) = \frac{\frac{\sqrt{2}\lambda}{k+1}}{1-\frac{k-1}{k+1}}=\frac{\sqrt{2}\lambda}{2}=\frac{\lambda}{\sqrt{2}}$

$\Rightarrow \quad \tan^2(\alpha+\beta)=\frac{\lambda^2}{2}=50$ [Given]

$\Rightarrow \quad \lambda = \pm 10$

**15. (b) :** Equation of plane $P$ passing through the points $(2, 1, 0)$, $(4, 1, 1)$ and $(5, 0, 1)$ is $x + y - 2z = 3$.

Let $(\alpha, \beta, \gamma)$ be the image of $R(2, 1, 6)$ w.r.t. plane $P$.

$\therefore \quad \frac{\alpha-2}{1}=\frac{\beta-1}{1}=\frac{\gamma-6}{-2}=\frac{-2(2+1-12-3)}{1+1+4}=4$

$\Rightarrow \quad (\alpha, \beta, \gamma) = (6, 5, -2)$

**16. (c) :** Given, $x^k + y^k = a^k$

Differentiating both sides w.r.t. $x$, we get

$kx^{k-1} + ky^{k-1}\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx}=-\left(\frac{x}{y}\right)^{k-1} \quad \Rightarrow \quad \frac{dy}{dx}+\left(\frac{x}{y}\right)^{k-1}=0$

Also, $\frac{dy}{dx}+\left(\frac{y}{x}\right)^{\frac{1}{3}}=0 \quad \therefore \quad k-1=-\frac{1}{3} \Rightarrow k=1-\frac{1}{3}=\frac{2}{3}$

**17. (b) :** Using Lagranges Mean Value Theorem for $x \in [-7, -1]$, we have

$\frac{f(-1)-f(-7)}{(-1+7)}\le 2 \Rightarrow \frac{f(-1)+3}{6}\le 2$ [Given $f(-7)=-3$]

$\Rightarrow f(-1) \le 9$

Also, using Lagranges Mean Value Theorem for $x \in [-7, 0]$, we have

$\frac{f(0)-f(-7)}{(0+7)}\le 2 \Rightarrow \frac{f(0)+3}{7}\le 2 \Rightarrow f(0)\le 11$

$\therefore \quad f(0) + f(-1) \le 20$

**18. (d) :** Given, $e^y\left(\frac{dy}{dx}-1\right)=e^x$ ...(i)

Put $e^y = t \Rightarrow e^y\frac{dy}{dx}=\frac{dt}{dx}$

$\therefore$ From (i), $\frac{dt}{dx}-t=e^x$, which is a linear differential equation

$\therefore$ I.F. $= e^{\int -1dx}=e^{-x}$

$\therefore$ Solution is $t(e^{-x}) = \int e^x\cdot e^{-x}dx+c$

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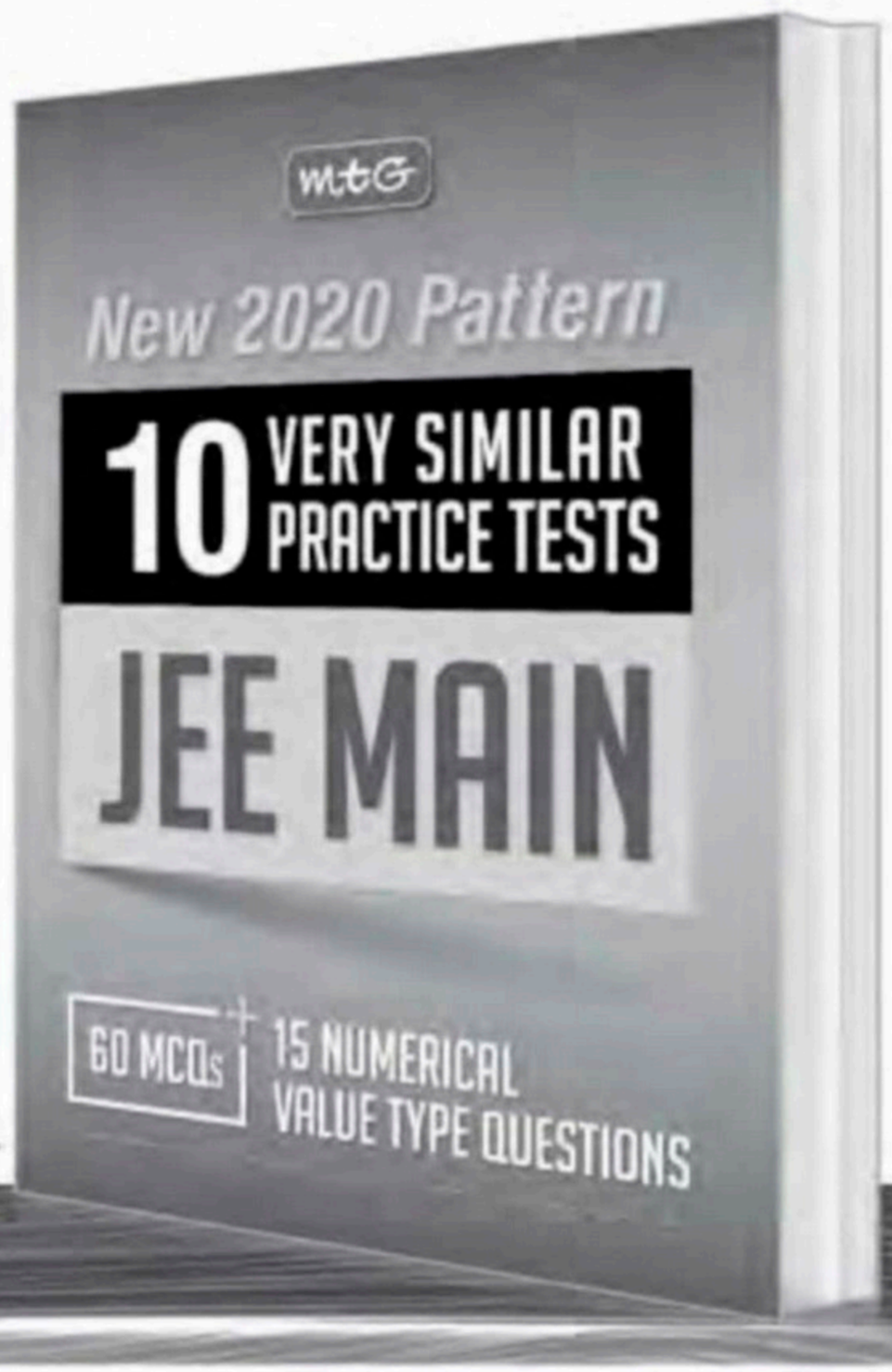


## HIGHLIGHTS

- 10 Very Similar Practice Tests as per the latest pattern of NTA JEE Main 2020 (60 MCQs+ 15 Numerical Value Type Questions)
- OMR sheet provided at the end of each test
- Detailed solutions of each practice test included

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$\Rightarrow e^{y-x} = x + c$ ...(ii)

Putting $x = 0, y = 0$ in (ii), we get $c = 1$

$\therefore \quad e^{y-x} = x + 1 \Rightarrow y = x + \log_e(x + 1)$

When $x = 1$, $y = 1 + \log_e(2)$

**19. (a) :** Let terms of the A.P. are $a - 2d$, $a - d$, $a$, $a + d$, $a + 2d$.

Given, sum of terms $= 25 \Rightarrow 5a = 25 \Rightarrow a = 5$

Also, product of terms $= 2520$ [Given]

$\Rightarrow (5 - 2d)(5 - d)\,5(5 + d)(5 + 2d) = 2520$

$\Rightarrow (25 - 4d^2)(25 - d^2) = 504$

$\Rightarrow 625 - 100d^2 - 25d^2 + 4d^4 = 504$

$\Rightarrow 4d^4 - 125d^2 + 121 = 0$

$\Rightarrow 4d^4 - 121d^2 - 4d^2 + 121 = 0$

$\Rightarrow (d^2 - 1)(4d^2 - 121) = 0 \Rightarrow d = \pm 1, d = \pm\dfrac{11}{2}$

Since, $d = \pm 1$ does not give $\dfrac{-1}{2}$ as one of the term of the A.P., so $d = \pm\dfrac{11}{2}$.

$\therefore$ A.P. is $-6, -\dfrac{1}{2}, 5, \dfrac{21}{2}, 16$.

Thus, greatest term is 16.

**20. (c) :** For non-trivial solution

$$\begin{vmatrix} 2 & 2a & a \\ 2 & 3b & b \\ 2 & 4c & c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$$

$\Rightarrow (3bc - 4bc) - (2ac - 4ac) + (2ab - 3ab) = 0$

$\Rightarrow -bc + 2ac - ab = 0 \Rightarrow ab + bc = 2ac \Rightarrow b = \dfrac{2ac}{a+c}$

$\Rightarrow a, b, c$ are in H.P. $\Rightarrow \dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in A.P.

**21. (36) :** We have, $\lim\limits_{x\to 2} \dfrac{3^x + 3^{3-x} - 12}{3^{\frac{-x}{2}} - 3^{1-x}}$ ...(i)

Putting $3^{\frac{x}{2}} = t$ in (i), we get

$$\lim_{t\to 3} \frac{t^2 + \frac{3^3}{t^2} - 12}{\frac{1}{t} - \frac{3}{t^2}} = \lim_{t\to 3} \frac{t^4 + 27 - 12t^2}{t - 3} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{t\to 3} \frac{4t^3 - 24t}{1} \qquad \text{[Applying L' Hospital's rule]}$$

$= 36$

**22. (18) :** Given, variance of first $n$ natural numbers $= 10$

$$\Rightarrow \frac{1^2 + 2^2 + \ldots\ldots + n^2}{n} - \left(\frac{1 + 2 + \ldots\ldots + n}{n}\right)^2 = 10$$

$$\Rightarrow \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = 10 \Rightarrow \frac{n^2 - 1}{12} = 10$$

$\Rightarrow n^2 - 1 = 120 \Rightarrow n = 11$

Now, var $(2, 4, 6, \ldots\ldots, 2m) = 16$ [Given]

$\Rightarrow$ var $(1, 2, \ldots\ldots, m) = \dfrac{16}{2^2} = 4$

$\Rightarrow m^2 - 1 = 48 \Rightarrow m = 7$

$\therefore \quad m + n = 18$

**23. (30) :** Let $(1 + x + x^2 + \ldots\ldots + x^{2n})(1 - x + x^2 - \ldots\ldots + x^{2n}) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{4n}x^{4n}$

Putting $x = 1$, we get

$(2n + 1)1 = a_0 + a_1 + a_2 + \ldots\ldots + a_{4n}$ ...(i)

Putting $x = -1$, we get

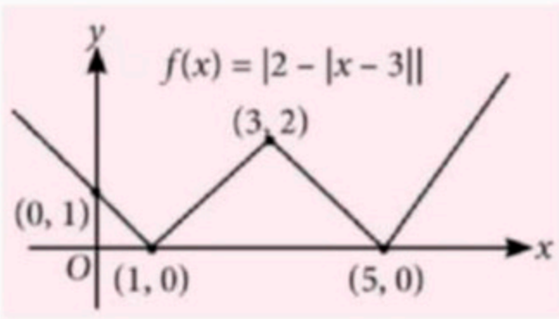
$1\,(2n + 1) = a_0 - a_1 + a_2 - \ldots\ldots + a_{4n}$ ...(ii)

Adding (i) and (ii), we get

$4n + 2 = 2(a_0 + a_2 + \ldots. + a_{4n}) = 2 \times 61$

$\Rightarrow 2n + 1 = 61 \Rightarrow n = 30$

**24. (3) :** Since $f(x)$ is not differentiable at $x = 1, 3, 5$.




So, $S = \{1, 3, 5\}$

$\therefore \quad \sum\limits_{x\in S} f(f(x)) = f(f(1)) + f(f(3)) + f(f(5))$

$= f(0) + f(2) + f(0)$ [$\because\ f(1) = f(5) = 0, f(3) = 2$]

$= 1 + 1 + 1$ [$\because\ f(0) = f(2) = 1$]

$= 3$

**25. (5) :** Since, ar$(\Delta PAB)$ = ar$(\Delta PAC)$ = ar$(\Delta PBC)$ [Given]

$\therefore$ $P$ will be centroid of $\Delta ABC$.

So, coordinates of $P$ are $\left(\dfrac{17}{6}, \dfrac{8}{3}\right)$.

Now, $PQ = \sqrt{\left(\dfrac{24}{6}\right)^2 + \left(\dfrac{9}{3}\right)^2} = 5$

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2020

Unit 8

## Limits, Continuity and Differentiability & Application of Derivatives

### DEFINITION OF LIMIT

- Let $y = f(x)$ be a function of $x$ and $x = a$ be any real number, then the number $l$ is said to be limit of $f(x)$ at $x = a$, if for any arbitrarily chosen positive number $\varepsilon$, however small but not zero, there exist a corresponding number $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $0 < |x - a| < \delta$.
- The meaning of $x \to a$ : Let $x$ be a variable quantity and $a$ be any real number. If $x$ assumes values nearer and nearer to $a$, then we say '$x$ tends to $a$', which does not mean $x = a$.

### Right and Left Hand Limit (R.H.L. and L.H.L.)

- Right hand limit means tendency of the function when we approach $x = a$ from the values which are just greater than $a$ and we write it, as

$$\lim_{x \to a^+} f(x) \text{ or } f(a^+)$$

To evaluate $\lim_{x \to a^+} f(x)$, we put $x = a + h$ in $f(x)$ to get $f(a + h)$ and take the limit as $h \to 0^+$.

- Left hand limit means tendency of function, when we approach $x = a$ from the values which are just less than $a$ and we write it, as

$$\lim_{x \to a^-} f(x) \text{ or } f(a^-)$$

To evaluate $\lim_{x \to a^-} f(x)$, we put $x = a - h$ in $f(x)$ to get $f(a - h)$ and take the limit as $h \to 0^+$.

- **Existence of the limit :** The $\lim_{x \to a} f(x)$ exists when both $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ are exists and equal.

**Note :** The limit of $f(x)$ as $x \to a$ does not exist if
(i) $f(a^+)$ and $f(a^-)$ exist, but $f(a^+) \neq f(a^-)$
(ii) either or both $f(a^+)$ and $f(a^-)$ does not exist.

### Indeterminate Forms

If the function $f(x)$ takes any one of the following forms

$$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, 1^\infty, 0^0, \infty^0 \text{ at } x = a,$$

then $f(x)$ is said to be indeterminate at $x = a$.

### Difference Between the Value and the Limit of a Function

$\lim_{x \to a} f(x)$ and $f(a)$ may not be same. In fact, one may be defined and not the other one.

For example, let $f(x) = \frac{x^2 - a^2}{x - a}$, the value of the function $f(x)$ at $x = a$ is 0/0, which is meaningless and known as indeterminate form.

But $\lim_{x \to a} f(x) = \lim_{x \to a} \frac{x^2 - a^2}{x - a} \left( \frac{0}{0}, \text{indeterminate form} \right)$

$= \lim_{x \to a} \frac{(x - a)(x + a)}{x - a} = \lim_{x \to a} (x + a)$, getting rid of indeterminate form.

$= 2a$, which is the limit of $f(x)$ at $x = a$.

By : **R. K. Tyagi**, Retd. Principal, HOD Maths, Samarth Shiksha Samiti, New Delhi

## L-Hospital's Rule

- Let $f(x)$ and $g(x)$ be two functions of $x$ such that $f(a) = 0$ and $g(a) = 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}, \text{ provided}$$

(i) both $f(x), g(x)$ are continuous and differentiable at $x = a$.

(ii) both $f'(x)$, $g'(x)$ are continuous at $x = a$ and $g'(a) \neq 0$.

**Note :** (i) Before applying L-Hospital's rule, check that the form of the function after substituting the limit is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ or not. If it is not, then make necessary change in the function to reduced it in this form.

(ii) If $f(a) = \infty$ and $g(a) = \infty$, then again $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$ provided the later limit exists.

(iii) If the later limit again take the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then again apply L-Hospital's rule on the later limit.

- **Sandwich theorem :** If $f(x)$, $g(x)$ and $h(x)$ are any three functions such that $f(x) \le g(x) \le h(x)$ $\forall\ x \in$ *nbd* of $x = a$ (not necessarily at $x = a$) and

$$\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=l \text{ (say), then } \lim_{x\to a} g(x)=l.$$

## Fundamental Theorems on Limits

Let $f(x)$ and $g(x)$ be two functions having limits $l$ and $m$ respectively when $x \to a$, where $l$, $m$ are finite and unique. Then

- $\lim_{x\to a} f(x)\pm g(x)=l\pm m$
- $\lim_{x\to a} f(x)\cdot g(x)=l\cdot m$
- $\lim_{x\to a} c\cdot f(x)=c\cdot l,\ c\in R$
- $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{l}{m},\ m\neq 0$
- $\lim_{x\to a} f(g(x))=f\left(\lim_{x\to a} g(x)\right)=f(m)$, provided $f$ is continuous at $g(x) = m$.
- $\lim_{x\to a}\log[f(x)]=\log\left(\lim_{x\to a} f(x)\right)$
- If $\lim_{x\to a} f(x)=+\infty$ or $-\infty$, then $\lim_{x\to a}\frac{1}{f(x)}=0$
- If $f(x)\le g(x)\ \forall\ x$, then $\lim_{x\to a} f(x)\le\lim_{x\to a} g(x)$
- $\lim_{x\to a}[f(x)+k]=\lim_{x\to a} f(x)+k,\ k\in R$

## Method of Calculating the Limit of the Form, $\lim_{x\to 0}(f(x))^{g(x)}$

- If the given limit is of the form $\lim_{x\to 0}(f(x))^{g(x)}$, then we express the given expression as power of $e$.

$$\text{i.e., } \lim_{x\to a}(f(x))^{g(x)}=e^{\lim_{x\to a} g(x)\cdot\log(f(x))}$$

$$=e^{\lim_{x\to a} g(x)\cdot\lim_{x\to a}\log(f(x))}=e^{\alpha\log\beta}=\beta^{\alpha},$$

where $\lim_{x\to a} g(x)=\alpha$ and $\lim_{x\to a} f(x)=\beta>0$.

**Note :** (i) If $f(x)$ is not throughout positive in the *nbd* of $x = a$, then $\lim_{x\to a}(f(x))^{g(x)}$ will not exist.

(ii) If $p$, $q$ are integers and $\lim_{x\to a} f(x)=l$, then $\lim_{x\to a}(f(x))^{p/q}=l^{p/q}$, where $(l)^{p/q}\in R$.

## SOME RESULTS ON LIMITS

### (A) Limits of Polynomials and Rational Functions

- $\lim_{x\to k}\frac{x^m-k^m}{x^n-k^n}=\frac{m}{n}k^{m-n}$
- $\lim_{x\to a}\frac{x^a-a^x}{x^x-a^a}=\frac{1-\log a}{1+\log a}$
- $\lim_{x\to\infty}\frac{a_0x^m+a_1x^{m-1}+\ldots+a_m x^0}{b_0x^n+b_1x^{n-1}+\ldots+b_n x^0}=\begin{cases}\frac{a_0}{b_0}, & m=n\\ 0, & m<n\\ \infty, & m>n,\ \frac{a_0}{b_0}>0\\ -\infty, & m>n,\ \frac{a_0}{b_0}<0\end{cases}$

### (B) Some Trigonometric Limits

- $\lim_{x\to 0}\frac{ax\pm\sin bx}{cx\pm\sin dx}=\frac{a\pm b}{c\pm d}$
- $\lim_{x\to 0}\frac{ax\pm b\sin cx}{dx\pm b\sin px}=\frac{a\pm bc}{d\pm bp}$
- $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}=\frac{m^2}{n^2}$
- $\lim_{x\to 0}\frac{1+\cos mx}{1+\cos nx}=1$
- $\lim_{x\to 0}\frac{1-\cos^n x}{1-\cos 2x}=\lim_{x\to\pi/4}\frac{1-\cos^n\left(x-\frac{\pi}{4}\right)}{1-\sin 2x}=\frac{n}{2^2}$
- $\lim_{x\to 0}\frac{\sin^2 ax\pm\sin^2 bx}{\sin^2 cx}=\frac{a^2\pm b^2}{c^2}$
- $\lim_{x\to 0}\frac{\sin^p mx}{x^p}=m^p$
- $\lim_{x\to 0}\frac{\sin^p mx}{\sin^p nx}=\left(\frac{m}{n}\right)^p$
- $\lim_{x\to 0}\sin mx=\lim_{x\to 0}\tan nx=0$
- $\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1$
- $\lim_{x\to\infty}\frac{\sin x}{x}=\lim_{x\to\infty}\frac{\cos x}{x}=0$
- $\lim_{x\to\infty}\frac{\sin(1/x)}{1/x}=1$

## (C) Limits of Logarithmic and Exponential Functions

- $\lim_{x\to\infty} a^x = \begin{cases} 0, & \text{if } 0 \le a < 1 \\ 1, & \text{if } a = 1 \\ \infty, & \text{if } a > 1 \end{cases}$ and does not exist if $a < 0$
- $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a,\ (a > 0)$
- $\lim_{x\to 0} \frac{e^{\lambda x} - 1}{x} = \lambda, (\lambda \neq 0)$
- $\lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1$
- $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e = \frac{1}{\log_e a}$

## (D) Limits Based on the Form $1^\infty$

- $\lim_{x\to 0} (1+ax)^{b/x} = e^{ab}$
- $\lim_{x\to\infty} \left(\frac{x+a}{x+b}\right)^{x+c} = e^{a-b}$
- $\lim_{n\to\infty} {}^nC_r \left(\frac{m}{n}\right)^r \left(1-\frac{m}{n}\right)^{n-r} = \frac{m^r e^{-m}}{r!}$

## (E) Some Expansion used to Evaluate the Limits

- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\infty$
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\infty$
- $a^x = 1 + x(\log a) + \frac{(x\log a)^2}{2!} + \frac{(x\log a)^3}{3!} + \ldots + \frac{(x\log a)^n}{n!} + \ldots\infty,\ a > 0,\ a \neq 1$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\infty$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\infty$
- $(1+x)^{1/x} = e \cdot \left(1 - \frac{x}{2} + \frac{11}{24}x^2 - \ldots\infty\right)$

## CONTINUITY

- **Continuity at a point :** A function $f(x)$ is said to be continuous at $x = a$, if $\lim_{x\to a} f(x) = f(a)$, *i.e.*,

  $\lim_{h\to 0^+} f(a-h) = \lim_{h\to 0^+} f(a+h) = f(a)$
- **Continuity in the open interval :** A function $f(x)$ is said to be continuous in an open interval $]a, b[$, if it is continuous at each and every point of the interval.
- **Continuity in the closed interval :** A function $f(x)$ is said to be continuous in a closed interval $[a, b]$, if
  (i) $f(x)$ is continuous at $x = a$, from the right of $x = a$, *i.e.*, $\lim_{x\to a^+} f(x) = f(a)$
  (ii) $f(x)$ is continuous at $x = b$, from the left of $x = b$, *i.e.*, $\lim_{x\to b^-} f(x) = f(b)$
  (iii) $f(x)$ is continuous at all points in the open interval $]a, b[$.

### Properties of Continuous Function

- Let $f(x)$ and $g(x)$ be two functions which are continuous at $x = a$, then
  (i) $f(x) \pm g(x)$ is continuous at $x = a$.
  (ii) $f(x) \cdot g(x)$ is continuous at $x = a$.
  (iii) $f(x)/g(x)$ is continuous at $x = a$, provided $g(a) \neq 0$.
  (iv) $c \cdot f(x)$ and $c \cdot g(x)$ both are continuous at $x = a$, provided $f(a)$ and $g(a)$ are finite quantities and $c \in R$.
- Let $f(x)$ is continuous at $x = a$ and $g(x)$ be continuous at $x = f(a)$, then composite function $g(f(x))$ is continuous at $x = a$.

### Discontinuity of a Function

A function is said to be discontinuous at a point $x = a$, if it is not continuous at that point. The point $x = a$ is called the point of discontinuity.

- The discontinuity may arise if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exists, but either of the two or both may not be equal to $f(a)$.
- If $f(x)$ is continuous and $g(x)$ is discontinuous at $x = a$, then the product *i.e.*, $h(x) = f(x)\ g(x)$ is not necessarily be discontinuous at $x = a$, but the sum or difference *i.e.*, $h(x) = f(x) \pm g(x)$ is always discontinuous at $x = a$.
- If $f(x)$ and $g(x)$ both are discontinuous at $x = a$, then it is not necessary that sum, product and difference function be discontinuous at $x = a$.

### Types of Discontinuity

- **Discontinuity of 1st kind :** The point $x = a$ is called the point of discontinuity of first kind, if both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist but are not equal.
- **Removable discontinuity :** The function $f(x)$ is said to have a removable discontinuity at $x = a$, if the limit of $f(x)$ at $x = a$ exist but is not equal to $f(a)$ *i.e.*, if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) \neq f(a)$.
- **Discontinuity of 2nd kind :** The point $x = a$ is called the point of discontinuity of the second kind, if either or both the limits $f(a^+)$ and $f(a^-)$ does not exist.

- **Piece-wise continuous or sectionally continuous function :** If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist finitely but not equal at a point $x = a$ of an interval $I$, then $\left|\lim_{x\to a^+} f(x) - \lim_{x\to a^-} f(x)\right|$ is called the jump of the function $f(x)$ at $x = a$.
- A function having a finite number of jumps in a given interval $I$ is called a piece-wise continuous function in this interval. For example, fractional part function and greatest integer function *i.e.*, $\{x\}$, $[x]$, are piece-wise continuous functions.

## DIFFERENTIABILITY OF A FUNCTION AT A POINT

- The right hand derivative of $f(x)$ at $x = a$ is denoted by $f'(a^+)$ and is defined as

$$\text{R.H.D} = f'(a^+) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}, \text{ provided the limit exist.}$$

- The left hand derivative of $f(x)$ at $x = a$ is denoted by $f'(a^-)$ and is defined as

$$\text{L.H.D} = f'(a^-) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}, \text{ provided the limit exist.}$$

- The function $f(x)$ is said to be differentiable at $x = a$ if $f'(a^+)$, $f'(a^-)$ *i.e.*, $Lf'(a)$, $Rf'(a)$ both exist finitely and are equal and their common value is called the derivative or differential coefficient at the point $x = a$.
- The function $f(x)$ is said to be non-differentiable at $x = a$, if
  (i) either or both $Rf'(a)$ and $Lf'(a)$ does not exist.
  (ii) both $Rf'(a)$ and $Lf'(a)$ exist but are not equal.

### Relation Between Differentiability and Continuity

- Differentiability implies continuity always, but if $f$ is continuous at $x = a$, then it is not necessary that $f$ is differentiable at $x = a$.

### Differentiability of Sum, Product, Quotient and Composition of Functions

- Let $f(x)$ and $g(x)$ are two differentiable function at $x = a$, then functions $f(x) \pm g(x)$, $f(x) \cdot g(x)$ will also be differentiable and if $g(a) \neq 0$, then $f(x)/g(x)$ will also be differentiable at $x = a$.
- If both $f(x)$ and $g(x)$ are non-differentiable at $x = a$, then the sum/difference function $h(x) = f(x) \pm g(x)$ and the product function $h(x) = f(x) \cdot g(x)$ may be differentiable at $x = a$.
- If $f(x)$ is not differentiable at $x = a$ and $g(x)$ is differentiable at $x = a$, then the product function $h(x) = f(x)g(x)$ may be differentiable at $x = a$, but sum/difference function $h(x) = f(x) \pm g(x)$ is always non-differentiable at $x = a$.
- The composition of differentiable function is a differentiable function.

### Algebra of Derivatives

If $u(x)$ and $v(x)$ are derivable function of $x$, then

- $\frac{d}{dx}(u(x) \pm v(x)) = \frac{d}{dx}(u(x)) \pm \frac{d}{dx}(v(x))$
- $\frac{d}{dx}(u(x) \cdot v(x)) = u(x)\frac{d}{dx}(v(x)) + v(x)\frac{d}{dx}(u(x))$
- $\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v(x)\frac{d}{dx}(u(x)) - u(x)\frac{d}{dx}(v(x))}{(v(x))^2}$

### Derivative of Composite Function (Chain Rule)

- If $y = f(u)$, $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
- If $y = f(t)$, $t = g(u)$, $u = h(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{du} \cdot \frac{du}{dx}$

### Derivative of an Implicit Function

When the variables $x$ and $y$ are connected by a relation of the form $f(x, y) = 0$ and it is not possible to express $y$ as a function of $x$, then $y$ is said to be an implicit function of $x$. To find $\frac{dy}{dx}$, differentiate each term w.r.t. $x$ and note that

$$\frac{d}{dx}(\phi(y)) = \frac{d\phi}{dy} \cdot \frac{dy}{dx}$$

### Standard Formulae of Differentiation

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(a^x) = a^x(\log_e a), a > 0$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(\log_e |x|) = \frac{1}{x}$
- $\frac{d}{dx}(\log_a |x|) = \frac{1}{x}\log_a e$

### Derivative of Trigonometric Functions

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$
- $\frac{d}{dx}(\text{cosec} x) = -\text{cosec} x \cdot \cot x$
- $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

### Second and Higher Order Derivative

The first order derivative of $y = f(x)$ w. r. t. $x$ is $y_1 = \frac{dy}{dx} = f'(x)$;

$y_2 = \frac{d^2y}{dx^2} = f''(x)$ is called the second order derivative

of $y$ w.r.t. $x$ and so on $y_n = \frac{d^n y}{dx^n} = f^n(x)$ is called the $n^{th}$ order derivative of $y$ w.r.t. $x$.

- If $y = x^n$, then $y_n = n!$
- If $y = e^{ax}$, then $y_n = a^n e^{ax}$
- If $y = \frac{1}{ax+b}$, then $y_n = \frac{(-1)^n a^n \cdot (n!)}{(ax+b)^{n+1}}$
- If $y = \log(ax + b)$, then $y_n = \frac{(-1)^{n-1}(n-1)!}{(ax+b)^n} \cdot a^n$
- If $y = \sin mx$, then $y_n = m^n \sin\left(n\frac{\pi}{2} + mx\right)$
- If $y = \frac{1}{(x-a)(x-b)} = \frac{1}{(b-a)}\left[\frac{1}{x-b} - \frac{1}{x-a}\right] a < b$

[Using partial fraction]

then $y_n = \frac{1}{(b-a)}(-1)^n n!\left[\frac{1}{(x-b)^{n+1}} - \frac{1}{(x-a)^{n+1}}\right]$

- If $y = \sin^3 x = \frac{1}{4}[3\sin x - \sin 3x]$, then

$$y_n = \frac{1}{4}\left[3\sin\left(\frac{n\pi}{2} + x\right) - 3^n \sin\left(\frac{n\pi}{2} + 3x\right)\right]$$

## APPLICATION OF DERIVATIVES

### Tangent and Normal

- **Slope of tangent :** Let $y = g(x)$ be a continuous curve (function) and $A(x_1, y_1)$ be any point on the curve, then slope of tangent at $A(x_1, y_1)$ is $\left(\frac{dy}{dx}\right)_{A(x_1,y_1)} = (g'(x))_{A(x_1,y_1)}$ and it is denoted by $m$ or $\tan\theta$

  **Note :** (i) If $\theta = 0°$, then tangent is parallel to $x$-axis.
  (ii) If $\theta = 90°$, then we can't define the slope at the said point but we write it as 1/0. Also, in this case tangent is parallel to $y$-axis.
- **Slope of normal :** Slope of normal at $A(x_1, y_1)$ = – 1/Slope of tangent at $A(x_1, y_1)$

**Equation of tangent and normal at $A(x_1, y_1)$ to the curve $y = g(x)$**

- Equation of tangent is

$$(y - y_1) = \left(\frac{dy}{dx}\right)_{A(x_1,y_1)} (x - x_1) = m(x - x_1)$$

- Equation of normal is

$$(y - y_1) = -\frac{1}{\left(\frac{dy}{dx}\right)_{A(x_1,y_1)}}(x - x_1)$$

*i.e.*, $(x - x_1) + m(y - y_1) = 0$

**Equation of tangent and normal, if curve are given in parametric form**

- Let $x = f(t), y = g(t)$ be the parametric equation of the curve, then slope of tangent at $A(x_1, y_1) = \frac{g'(t_1)}{f'(t_1)}$, here $A(x_1, y_1) = A\ (f(t_1), g(t_1))$.
- Equation of tangent is $y - g(t_1) = \frac{g'(t_1)}{f'(t_1)}(x - f(t_1))$
- Equation of normal is

$(y - g(t_1))\cdot g'(t_1) + (x - f(t_1))\cdot f'(t_1) = 0$

### Point of Intersection and Angle of Intersection of Two Curves

If $y = f(x)$ and $y = g(x)$ be two curves, then

- For point of intersection, solve $f(x) = g(x)$.
- The angle of intersection of the curve is the angle between the tangents to the two curves at their point of intersection.
- If $m_1$ and $m_2$ be the slopes of tangents to the curves at their point of intersection and $\theta$ be the angle of their intersection, then $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$ ... (*)
- If two curves touches each other, then $\theta = 0°$ and from (*), we get $m_1 = m_2$.
- If two curves are orthogonal (perpendicular to each other), then $\theta = 90°$ and from (*), we get $1 + m_1 m_2 = 0 \Rightarrow m_1 m_2 = -1$.

### Monotonicity of a Function

Let $y = f(x)$ be a continuous function in $(a, b)$, then $f(x)$ is said to be monotonic if the following results holds:

- Increasing if $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$
- Strictly increasing if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- Decreasing if $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$
- Strictly decreasing if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ where $x_1, x_2 \in (a, b)$

**Note :** (i) It may be possible that a function is neither increasing nor decreasing in a given interval.
(ii) On an interval $I$, $f(x)$ is
strictly increasing if $f'(x) > 0$
strictly decreasing if $f'(x) < 0$

### Rolle's Theorem

Let $y = f(x)$ be a function defined on $[a, b]$, then the Rolle's theorem will be applicable on $f(x)$ if the following three conditions hold.

- $f(x)$ must be continuous on $[a, b]$.
- $f(x)$ must be differentiable in $]a, b[$.
- The value of $f(x)$ at $x = a$, $x = b$ are equal *i.e.*, $f(a) = f(b)$.

If the above conditions satisfied, then there exist at least one real value $c \in ]a, b[$ such that $f'(c) = 0$.

### Lagrange's Mean Value Theorem (L.M.V. Theorem)

Let $y = f(x)$ be a function defined on $[a, b]$, then L.M.V. theorem will be applicable on $f(x)$ if first two conditions of Rolle's theorem are satisfied. If these conditions satisfied, then there exist at least one value $c \in ]a, b[$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

## PROBLEMS

### Single Option Correct Type

**1.** Let $g$ is continuous on $[\alpha, \beta], \alpha > 0$ and differentiable on an interval $]\alpha, \beta[$. If $\frac{g(\alpha)}{\alpha}=\frac{g(\beta)}{\beta}$, then there exist a number $x_0 \in ]\alpha, \beta[$ such that

(a) $g'(x_0) = x_0^2\, g(x_0)$ (b) $x_0\, g'(x_0) = -g(x_0)$

(c) $g'(x_0) + x_0\, g(x_0) = 0$ (d) $g(x_0) = x_0\, g'(x_0)$

**2.** Let $g(x)$ be a polynomial of degree 4 such that at $x = 2$, $x = 1/2$ it attains extreme values and $\lim_{x\to 0}\left(1+\frac{g(x)}{x^2}\right)=3$, then the value of $g(3)$ is

(a) 0 (b) 3 (c) 6 (d) 9

**3.** If $f(x)=\frac{1}{(x-4)(x-5)}$ and $g(x)=\frac{1}{x^2}$, then number of points of discontinuity of $f(g(x))$ are

(a) 3 (b) 4 (c) 5 (d) 2

**4.** The graph of the function $y = g(x)$ has a unique tangent at a point $(x_0, 0)$, through which the graph passes, then evaluate $\lim_{x\to x_0}\frac{\log(1+9g(x))}{4g(x)}$

(a) 9/4 (b) 4/9 (c) −9/4 (d) − 4/9

**5.** For a curve $y = g(x)$, $g''(x) = 3x^2 - 2x$ at each point $(x, y)$. Also, when $y(-2) = 0$, the angle of inclination of the tangent to the curve with the positive direction of $x$-axis is 45°. If $g(x) = ax^4 + bx^3 + cx^2 + dx + e$, then $12(a - b + c - d + e)$ is

(a) −83 (b) 99 (c) 83 (d) 95

**6.** If $y = 4x - 6$ is a tangent to the curve $y^2 = ax^4 + b$ at (3, 6), then

(a) $a=\frac{4}{9}, b=\frac{-4}{9}$ (b) $a=0, b=\frac{4}{9}$

(c) $a=-\frac{4}{9}, b=-\frac{4}{9}$ (d) $a=\frac{4}{9}, b=0$

**7.** If normal to the curve $y = f(x)$ at $x = 0$ be given by the equation $9x - y + 3 = 0$, then the value of $\lim_{x\to 0} x^2\left\{f(x^2)-6f(4x^2)+5f(6x^2)\right\}^{-1}$ is

(a) −9/7 (b) −9 (c) −1/3 (d) −7/9

**8.** If the function $f(x) = ax^3 + bx^2 + 26x - 24$ satisfies conditions of Rolle's theorem in [2, 4] and $f'\left(3+\frac{1}{\sqrt{3}}\right)=0$, then values of $a$ and $b$ are respectively

(a) 1, −9 (b) −1, 9 (c) 3, −1 (d) −1, 3

**9.** If $f(x) = ax^3 - 9x^2 + 9x + 3$ is strictly increasing $\forall\, x \in R$, then

(a) $a \le 3$ (b) $a > 3$

(c) $a < 3$ (d) None of these

**10.** If $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right)\forall\, x, y\in R$ and $xy = 1$ and $\lim_{x\to 0}\frac{f(x)}{x}=2$, then value of $\frac{f\left(\frac{1}{\sqrt{3}}\right)}{f'(-3)}\left(\frac{1212}{\pi}\right)$ equals

(a) 2200 (b) 2020 (c) 2019 (d) 2021

**11.** Which one of the following function is differentiable in (−1, 2)?

(a) $\int_x^{2x}(\log t)^2\, dt$ (b) $\int_x^{2x}\frac{\sin t}{t}\, dt$

(c) $\int_0^{x}\frac{1-t+t^2}{1+t+t^2}\, dt$ (d) None of these

**12.** Let $f(x)=\begin{cases}\frac{a|x^2-x-2|}{2+x-x^2}, & x<2\\ \lambda, & x=2,\\ \frac{x-[x]}{x-2}, & x>2\end{cases}$

where $[\cdot]$ denotes the greatest integer function. If $f(x)$ is continuous at $x = 2$, then

(a) $a = \lambda = 1$ (b) $a = \lambda = 2$

(c) $a = 0, \lambda = 1$ (d) $a = 1, \lambda = 0$

**13.** The equation of tangent to the curve $\int_{x^2}^{x^3}\frac{dt}{\sqrt{1+t^2}}$ at $x = 1$ is

(a) $y=\sqrt{2}x-1$ (b) $y=\sqrt{2}x+1$

(c) $\sqrt{2}y=x-1$ (d) $\sqrt{2}y=x+1$

**14.** If $f''(x)$ is a positive function $\forall\ x \in R$, $f'(3) = 0$ and $g(x) = f(\tan^2 x - 2\tan x + 4)$ for $0 < x < \pi/2$, then the interval in which $g(x)$ increasing is

(a) $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ (b) $\left(0, \frac{\pi}{4}\right)$ (c) $\left(0, \frac{\pi}{3}\right)$ (d) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

**15.** For $x \in R$, $f(x) = (\log 2 - \sin x)$ and $g(x) = f(f(x))$, then

(a) $g$ is not differentiable at $x = 0$

(b) $g'(0) = -\cos(\log 2)$

(c) $g$ is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$

(d) $g'(0) = \cos(\log 2)$

## More Than One Option(s) Correct Type

**16.** Let $f(x)=\begin{cases} x^2+2, & \forall\ x<0 \\ 4, & \forall\ x=0 \\ x+2, & \forall\ x>0 \end{cases}$, then

(a) $f(x)$ is non-differentiable at $x = 0$

(b) $f(x)$ is strictly decreasing on the left of 0

(c) $f'(x)$ is strictly increasing on the left of 0

(d) $f(x)$ is strictly increasing on the right of 0

**17.** Let $f(x) = ax^3 + bx^2 + cx + d$ have extrema at $x = \alpha, \beta$ such that $\alpha\cdot\beta < 0$, $f(\alpha)\, f(\beta) < 0$ and $d$ is positive, then $f(x) = 0$ has

(a) all distinct real roots

(b) one negative root, if $f(\beta) > 0$ and $\alpha < 0$

(c) one positive root, if $f(\alpha) < 0$ and $\beta < 0$

(d) $f(x)$ can be factorised into three linear factors

**18.** Let $f(x)=\begin{cases} x^2(\text{sgn}[x])+\{x\}, & 0\le x<2 \\ \sin x+|x-3|, & 2\le x<4 \end{cases}$, where $[\cdot]$ and $\{\cdot\}$ represent the greatest integer and the fractional part functions respectively, then

(a) $f(x)$ is differentiable at $x = 1$

(b) $f(x)$ is non-differentiable at $x = 2$

(c) $f(x)$ is continuous at $x = 1$

(d) $f(x)$ is discontinuous at $x = 2$

**19.** If $f(x - y)$, $f(x) \times f(y)$, $f(x + y)$ are in A.P. for all $x, y$ and $f(0) \neq 0$, then

(a) $f(5) = f(-5)$

(b) $f'(x)$ is an odd function

(c) $f(x)$ is an even function

(d) $f'(-2) + f'(2) = 0$

**20.** Let $\alpha = \min(x^2 + 2x + 3), \forall\ x \in R$, $\beta = \lim\limits_{\theta\to 0} \frac{1-\cos\theta}{\theta^2}$, then

(a) the minimum value of $\alpha$ is 2

(b) $\beta = 1/2$

(c) $\sum\limits_{r=0}^{n} \alpha^r \beta^{n-r} = \frac{4^{n+1}-1}{3\cdot 2^n}$

(d) $x = -1$ is point of minima for $\alpha$

**21.** Let $f(x) = x^2 + xg'(1) + g''(2)$ and $g(x) = f(1)x^2 + xf'(x) + f''(x)$, then

(a) $g'(1) = -3$ (b) $g''(2) = 0$

(c) $f(x) = x^2 - 3x$ (d) $g(x) = -3x + 2$

**22.** If $\lim\limits_{x\to 1} \frac{a\sin(x-1)+b\cos(x-1)+8}{x^2-1} = 4$, then

(a) $a + 2b = -8$ (b) $2a + b = 8$

(c) $a = 8$ (d) $b = -8$

## Comprehension Type

### Paragraph for Q. No. 23 to 25

If $y = f(x)$ be a differential function of $x$ such that its derivative up to $n^{\text{th}}$ order exist and its $n^{\text{th}}$ order derivative is denoted by $y_n, \frac{d^n y}{dx^n}$ or $D^n y$.

Also, $\frac{d^n y}{dx^n} = \lim\limits_{h\to 0} \frac{f^{n-1}(x+h) - f^{n-1}(x)}{h}$

On the basis of above information, answer the following questions :

**23.** If $y = e^{4x+5}$, then $y_n(0)$ is

(a) 1 (b) $4^n e^5$ (c) $4^n$ (d) $4^n e^5 \cdot n!$

**24.** If $y=\frac{1}{2-3x}$, then value of $y_n(1)$ is

(a) 0 (b) $(-1)^n 3^n$

(c) $(-1)^n 3^n\, n!$ (d) $-n!\,(-3)^n$

**25.** If $y=\frac{1}{x^2-7x+12}+\sin x+\cos x$, then $y_n$ equals

(a) $(-1)^n n!\left[\frac{1}{(x-4)^{n+1}}-\frac{1}{(x-3)^{n+1}}\right] + \sqrt{2}\sin\left(n\frac{\pi}{2}+\overline{x+\frac{\pi}{4}}\right)$

(b) $(-1)^n n! + \sqrt{2}\sin\left(\overline{x+\frac{\pi}{4}}\right)$

(c) $(-1)^n n!\left(\frac{1}{(x-4)^n}-\frac{1}{(x-3)^n}\right)+\frac{\sqrt{2}\sin\left(n\pi+\overline{x+\frac{\pi}{4}}\right)}{n!}$

(d) None of these

## Matrix Match Type

**26.** Match the following :

| | Column-I | | Column-II |
|---|---|---|---|
| P. | The slope of the curve $3y^2 = ax^2 + b$ at $(1, 2)$ is 1, then | 1. | $a - b = 0$ |
| Q. | If $(a, b)$ lies on the curve $4y^2 = x^3$, where normal to the curve cuts equal intercept on both axes, then | 2. | $5a + 2b = 0$ |
| R. | If the tangent at any point $(1, 2)$ on the curve $y = ax^2 + bx + \frac{7}{2}$ is parallel to the normal $(-2, 2)$ on the curve $y = x^2 + 6x + 10$, then | 3. | $a - b = \frac{16}{27}$ |
| | | 4. | $a + \lvert b\rvert = \frac{80}{27}$ |
| | | 5. | $a - b = \frac{7}{2}$ |

**27.** Match the following :

| | Column-I | | Column-II |
|---|---|---|---|
| P. | If $f(x) = \left(\frac{\lvert x\rvert}{\lvert x\rvert+2}\right)^{-x}$, then | 1. | $\lim\limits_{x\to-\infty} f(x) = e^{-2}$ |
| Q. | If $f(x) = \frac{(1+x)^{1/x} - e}{x}$, then | 2. | $\lim\limits_{x\to0} f(x) = -\frac{e}{2}$ |
| R. | If $f(x) = \left(\frac{1+7x^2}{1+5x^2}\right)^{\frac{1}{x^2}}$, then | 3. | $\lim\limits_{x\to\infty} f(x) = e^2$ |
| | | 4. | $\lim\limits_{x\to0} f(x) = e^2$ |
| | | 5. | $\lim\limits_{x\to0} f(x) < -1$ |

## Numerical Value Type

**28.** If $f: [-1, 1] \to R$ be a continuous function satisfying $f(2x^2 - 1) = (x^3 + x)f(x)$, then $\lim\limits_{x\to0} \frac{f(\cos x)}{\sin x}$ is equal to

**29.** If $f(x) = \int_0^x f(t)dt$ and $f(x^3) = x^3(1+x)$, then the value of $f(27) = n$. Then unit place of $(49216)^{n^2}$ is

**30.** Let $f(x)$ be a function such that $\lim\limits_{x\to0} \frac{f(x)}{x} = 1$. If $\lim\limits_{x\to0} \frac{x(1+a\cos x) - b\sin x}{(f(x))^3} = 1$, then $\left[\frac{a}{b}\right]$, where [.] is greatest integer function, is

## SOLUTIONS

**1. (d) :** Let $h(x) = \frac{g(x)}{x}$, as $g(x)$ and $x$ are continuous and differentiable, thus $h(x)$ is so.

Now, $h(\alpha) = \frac{g(\alpha)}{\alpha}$ and $h(\beta) = \frac{g(\beta)}{\beta}$ but $\frac{g(\alpha)}{\alpha} = \frac{g(\beta)}{\beta}$ (Given)

$\therefore$ $h(\alpha) = h(\beta) \Rightarrow$ Rolle's theorem is applicable for $h(x)$.

Therefore, there exist a number $x_0 \in\ ]\alpha, \beta[$ such that $h'(x_0) = 0$.

Now, $h'(x) = \frac{xg'(x) - g(x)\cdot1}{x^2}$

$\therefore\ h'(x_0) = \frac{x_0g'(x_0) - g(x_0)}{x_0^2} \therefore h'(x_0) = 0 \Rightarrow x_0g'(x_0) = g(x_0)$

**2. (d) :** As $g(x)$ having degree 4

So, let $g(x) = ax^4 + bx^3 + cx^2 + dx + e$ ...(i)

Now, $\lim\limits_{x\to0}\left(1 + \frac{g(x)}{x^2}\right) = 3 \Rightarrow \lim\limits_{x\to0}\left(\frac{x^2 + g(x)}{x^2}\right) = 3$

$\Rightarrow$ $g(0) = 0$ (a finite quantity, using 0/0 form)

$\Rightarrow$ $e = 0$ ...(ii)

Again, $\lim\limits_{x\to0}\left(\frac{2x + g'(x)}{2x}\right) = 3$

$\Rightarrow$ $g'(0) = 0$ (a finite quantity, using 0/0 form)

$\Rightarrow$ $d = 0$ ...(iii)

Again, $\lim\limits_{x\to0}\left(\frac{2 + g''(x)}{2}\right) = 3 \Rightarrow g''(0) = 4$

(using hit and trial method for $g''(x)$, we get

L.H.S. = R.H.S.) $\Rightarrow$ $2c = 4 \Rightarrow c = 2$ ..(iv)

From (i), (ii), (iii) and (iv), we get

$g(x) = ax^4 + bx^3 + 2x^2 \Rightarrow g'(x) = 4ax^3 + 3bx^2 + 4x$

$\because$ $x = 2, 1/2$ are extremum points.

$\therefore$ $g'(2) = 0, g'(1/2) = 0$

$\Rightarrow$ $8a + 3b + 2 = 0$ and $2a + 3b + 8 = 0$

On solving these equations, we get $a = 1,\ b = -\frac{10}{3}$

$\therefore$ $g(x) = x^4 - \frac{10}{3}x^3 + 2x^2 \Rightarrow g(3) = 81 - 90 + 18 = 9$

**3. (c)**

**4. (a) :** $\lim\limits_{x\to x_0} \frac{\log(1+9g(x))}{4g(x)} = \lim\limits_{x\to x_0} \frac{\log(1+9g(x))}{9g(x)} \frac{9}{4}$

$= \frac{9}{4} \lim\limits_{x\to x_0} \frac{\left(9g(x) - \frac{9^2g^2(x)}{2} + \frac{9^3g^3(x)}{3} - \ldots\infty\right)}{9g(x)} = \frac{9}{4}$

[Given, $g(x_0) = 0$]

**5.** **(c) :** $\because\ g''(x) = 3x^2 - 2x$

$\therefore\quad g'(x) = x^3 - x^2 + c_1, \Rightarrow 1 = -8 - 4 + c_1$

$\Rightarrow\quad c_1 = 13$ (Given, $g'(-2) = \tan 45° = 1$)

$\therefore\quad g'(x) = x^3 - x^2 + 13 \Rightarrow g(x) = \dfrac{x^4}{4} - \dfrac{x^3}{3} + 13x + c_2$

$\Rightarrow\quad 0 = 4 + \dfrac{8}{3} - 26 + c_2$ (Using $g(-2) = 0$)

$\Rightarrow c_2 = \dfrac{58}{3}$

$\therefore\quad g(x) = \dfrac{x^4}{4} - \dfrac{x^3}{3} + 13x + \dfrac{58}{3} = ax^4 + bx^3 + cx^2 + dx + e$

$\Rightarrow\quad a = 1/4,\ b = -1/3,\ c = 0,\ d = 13$ and $e = \dfrac{58}{3}$

$\therefore\quad a - b + c - d + e = \dfrac{1}{4} + \dfrac{1}{3} - 13 + \dfrac{58}{3} = \dfrac{83}{12}$

$\Rightarrow\quad 12(a - b + c - d + e) = 83$

**6.** **(d)**

**7.** **(a) :** Given, $y = f(x) \Rightarrow \dfrac{dy}{dx} = f'(x)$

$\therefore\quad \left(\dfrac{dy}{dx}\right)_{x=0} = f'(0)$

$\therefore\quad$ Slope of normal $= -\dfrac{1}{f'(0)} = 9 \Rightarrow f'(0) = -\dfrac{1}{9}$

Now, $\lim\limits_{x\to 0} \dfrac{x^2}{f(x^2) - 6f(4x^2) + 5f(6x^2)}$

$= \lim\limits_{x\to 0} \dfrac{x}{f(x) - 6f(4x) + 5f(6x)}$ [Replacing $x^2$ by $x$]

$= \lim\limits_{x\to 0} \dfrac{x}{(f(x) - f(0)) - 6(f(4x) - f(0)) + 5(f(6x) - f(0))}$

$= \lim\limits_{x\to 0} \dfrac{1}{\dfrac{f(x) - f(0)}{x - 0} - 6\dfrac{(f(4x) - f(0))}{x - 0} + 5\dfrac{(f(6x) - f(0))}{x - 0}}$

$= \lim\limits_{x\to 0} \dfrac{1}{\dfrac{f(x) - f(0)}{x - 0} - 6\times 4\left(\dfrac{f(4x) - f(0)}{4x - 0}\right) + 5\times 6\left(\dfrac{f(6x) - f(0)}{6x - 0}\right)}$

$= \dfrac{1}{f'(0) - 24f'(0) + 30f'(0)} = \dfrac{1}{7f'(0)} = \dfrac{1}{7} \times (-9) = -\dfrac{9}{7}$

**8.** **(a) :** $\because\ f(x) = ax^3 + bx^2 + 26x - 24$ satisfies Rolle's theorem.

$\therefore\quad f(2) = f(4) \Rightarrow 14a + 3b + 13 = 0$ ... (i)

and $f'(x) = 3ax^2 + 2bx + 26$

Now, $f'\left(3 + \dfrac{1}{\sqrt{3}}\right) = 0$

$\Rightarrow 3a\left(3 + \dfrac{1}{\sqrt{3}}\right)^2 + 2b\left(3 + \dfrac{1}{\sqrt{3}}\right) + 26 = 0$

$\Rightarrow 3a\left(9 + \dfrac{1}{3} + \dfrac{6}{\sqrt{3}}\right) + 2b\left(3 + \dfrac{1}{\sqrt{3}}\right) + 26 = 0$

$\Rightarrow a(28 + 6\sqrt{3}) + 2b\left(3 + \dfrac{1}{\sqrt{3}}\right) + 26 = 0$ ... (ii)

Solving (i) and (ii), we get

$a\left(\dfrac{26}{\sqrt{3}}\right) + 26\left(-\dfrac{1}{\sqrt{3}}\right) = 0 \Rightarrow a = 1,\ b = -9$

**9.** **(b) :** $\because\ f(x) = ax^3 - 9x^2 + 9x + 3$

$\Rightarrow\quad f'(x) = 3ax^2 - 18x + 9$

For strictly increasing function, $f'(x) > 0$

$\Rightarrow\quad 3ax^2 - 18x + 9 > 0 \Rightarrow ax^2 - 6x + 3 > 0$

For $a > 0,\ D < 0$

$\Rightarrow\quad (-6)^2 - 4(a)(3) < 0 \Rightarrow a > 3$

**10.** **(b)**

**11.** **(c) :** (a) Let $f(x) = \int\limits_x^{2x} (\log t)^2\, dt$

$\Rightarrow\quad f'(x) = 2(\log 2x)^2 - (\log x)^2$

$\because\quad f'(x)$ does not exist in $(-1, 0] \subset (-1, 2)$

So, $f(x)$ is not differentiable in $(-1, 2)$

(b) Let $g(x) = \int\limits_x^{2x} \dfrac{\sin t}{t}\, dt \Rightarrow g'(x) = 2\left(\dfrac{\sin 2x}{2x}\right) - \dfrac{\sin x}{x}$,

which does not exist at $x = 0$

$\therefore\quad g(x)$ is not differentiable in $(-1, 2)$

(c) Let $h(x) = \int\limits_0^x \dfrac{1 - t + t^2}{1 + t + t^2}\, dt \Rightarrow h'(x) = \dfrac{1 - x + x^2}{1 + x + x^2}$

and $1 + x + x^2 = \left(x + \dfrac{1}{2}\right)^2 + \dfrac{3}{4} > 0$

So, $h'(x)$ is defined $\forall\ x \in (-1, 2)$, thus $h(x)$ is differentiable in $(-1, 2)$.

**12.** **(a) :**

$$f(x) = \begin{cases} \dfrac{-a(x^2 - x - 2)}{2 + x - x^2}, & -1 < x < 2 \\ \lambda, & x = 2 \\ \dfrac{x - [x]}{x - 2}, & x > 2 \end{cases} = \begin{cases} a, & -1 < x < 2 \\ \lambda, & x = 2 \\ \dfrac{x - [x]}{x - 2}, & x > 2 \end{cases}$$

Now, L.H.L. $= \lim\limits_{x\to 2^-} f(x) = \lim\limits_{h\to 0} f(2 - h) = \lim\limits_{h\to 0} a = a$

R.H.L. $= \lim\limits_{x\to 2^+} f(x) = \lim\limits_{h\to 0} f(2 + h)$

$= \lim\limits_{h\to 0} \dfrac{(2 + h) - [2 + h]}{2 + h - 2} = \lim\limits_{h\to 0} \dfrac{2 + h - 2}{2 + h - 2} = 1$

Also, the value of the function, $f(2) = \lambda$

$\because\quad f(x)$ is continuous at $x = 2$.

$\therefore\quad$ L.H.L. = R.H.L. $= f(2) \Rightarrow a = \lambda = 1$

**13. (c) :** Let $f(x)=y=\int_{x^2}^{x^3}\frac{dt}{\sqrt{1+t^2}}$

$\because$ Tangent passes through $x=1$

$\therefore\quad y(1)=\int_{1}^{1}\frac{dt}{\sqrt{1+t^2}}=0\Rightarrow$ Point on the curve is (1, 0).

Again, $f'(x)=\frac{3x^2}{\sqrt{1+x^6}}-\frac{2x}{\sqrt{1+x^4}}$

$\therefore\ f'(1)=\frac{3}{\sqrt{2}}-\sqrt{2}=\frac{1}{\sqrt{2}}$

Now, required equation of tangent, which passes through (1, 0) and having slope $\frac{1}{\sqrt{2}}$, is $\sqrt{2}y=x-1$.

**14. (d) :** Given, $g(x)=f(\tan^2x-2\tan x+4)$

$\Rightarrow\quad g'(x)=f'(\tan^2x-2\tan x+4)\cdot(2\tan x\sec^2x-2\sec^2x)$

$=2\sec^2x(\tan x-1)f'(\tan^2x-2\tan x+4)$

Now, as $f''(x)$ is positive *i.e.*, $f''(x)>0$

$\Rightarrow$ $f'(x)$ is strictly increasing function.

$\therefore$ So, $f'((\tan x-1)^2+3)>f'(3)=0$

$\forall\ x\in\left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

Also, $(\tan x-1)>0\ \forall\ x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

$\therefore$ $g(x)$ is increasing in the interval $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$.

**15. (d) :** Given, $f(x)=(\log 2-\sin x)$

Also, $g(x)=f(f(x))=f(\log 2-\sin x)$

$=\log 2-\sin(\log 2-\sin x)$

$=\log 2+(-\sin(\log 2-\sin x))$

As, $g(x)$ is sum of two differentiable functions, so, $g$ is differentiable.

Now, $g'(x)=0-\cos(\log 2-\sin x)(-\cos x)$

$=\cos x\cdot\cos(\log 2-\sin x)$

$\therefore\quad g'(0)=\cos 0\cdot\cos(\log 2-0)=\cos(\log 2)$

**16. (a, b, c, d)**

**17. (a, b, c, d) :** Since, $f(x)$ has extrema at $x=\alpha,\beta$.

$\therefore\quad f'(x)=0$ at $x=\alpha,\beta$

$\because\quad \alpha\beta<0$ therefore $\alpha<0,\beta>0$ or $\alpha>0,\beta<0$

Now, when $f(\beta)>0$ and $\alpha<0$, then $f(\alpha)<0$ and $\beta>0$.

Also, $f(0)=d>0$

$\therefore\quad f(x)=0$ has one negative root between $\alpha$ and $\beta$.

Also, when $f(\alpha)<0$ and $\beta<0$, then $f(\beta)>0$ and $\alpha>0$.

$\Rightarrow\quad f(x)=0$ has one positive root.

Clearly, $f(x)=0$ has three distinct real roots.

$\therefore\quad f(x)$ can be factorised into three linear factors.

**18. (b, c, d) :** $\because\quad f(x)=\begin{cases}x^2(\text{sgn}[x])+\{x\}, & 0\le x<2\\ \sin x+|x-3|, & 2\le x<4\end{cases}$

Continuity for $x=1$ :

$\lim_{x\to1^+}f(x)=\lim_{x\to1^+}(x^2(\text{sgn}[x])+\{x\})=1+0=1$

$\lim_{x\to1^-}f(x)=\lim_{x\to1^-}(x^2(\text{sgn}[x])+\{x\})=1^2\cdot\text{sgn}(0)+1=1$

and $f(1)=1+\{1\}=1+0=1$

$\therefore\quad \lim_{x\to1^+}f(x)=\lim_{x\to1^-}f(x)=f(1)$, so $f$ is continuous at $x=1$

For differentiability at $x=1$ :

$$f'(1^+)=\lim_{h\to0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to0}\frac{(1+h)^2\,\text{sgn}[1+h]+\{1+h\}-1}{h}$$

$$=\lim_{h\to0}\frac{(1+h)^2+h-1}{h}=\lim_{h\to0}\frac{h^2+3h}{h}=3$$

$$f'(1^-)=\lim_{h\to0}\frac{f(1-h)-f(1)}{-h}$$

$$=\lim_{h\to0}\frac{(1-h)^2\,\text{sgn}[1-h]+\{1-h\}-1}{-h}$$

$$=\lim_{h\to0}\frac{(1-h)^2\cdot0+(1-h)-1}{-h}=\lim_{h\to0}\frac{-h}{-h}=1$$

$\because\quad f'(1^+)\ne f'(1^-)\Rightarrow f(x)$ is not differentiable at $x=1$

Again, for $x=2$,

$\lim_{x\to2^+}f(x)=\lim_{x\to2}(\sin x+|x-3|)=\sin 2+1$

and $\lim_{x\to2^-}f(x)=\lim_{x\to2}(x^2\,\text{sgn}[x]+\{x\})=4\times1+1=5$

$\therefore\quad \lim_{x\to2^+}f(x)\ne\lim_{x\to2^-}f(x)$

$\Rightarrow$ $f(x)$ is discontinuous at $x=2$

$\therefore\quad f(x)$ cannot be differentiable at $x=2$

**19. (a, b, c, d) :** Since, $f(x-y), f(x)\cdot f(y)$ and $f(x+y)\in$ A.P.

$\Rightarrow\quad f(x-y)+f(x+y)=2f(x)f(y)$ ... (*)

Putting $x=y=0$ in (*), we get

$2f(0)=2(f(0))^2\Rightarrow f(0)=1$ (As, $f(0)\ne0$) ...(i)

Again, putting $x=0$ and $y=x$ in (*), we have

$f(-x)+f(x)=2f(0)f(x)$

$\Rightarrow\quad f(-x)=f(x)$ (Using (i)) ... (**)

$\therefore\quad f(x)$ is an even function and $f(-5)=f(5)$

Now, differentiating (**) with respect to $x$, we get

$-f'(-x)=f'(x)\Rightarrow f'(-x)=-f'(x)$

$\therefore\quad f'$ is an odd function and $f'(-2)+f'(2)=0$

**20. (a, b, c, d)**

**21. (a, b, c, d) :** Let $g'(1) = \alpha$ and $g''(2) = \beta$. ...(i)

Then, $f(x) = x^2 + \alpha x + \beta \Rightarrow f(1) = 1 + \alpha + \beta$,

$f'(x) = 2x + \alpha$ and $f''(x) = 2$ ...(*)

$\therefore \quad g(x) = (1 + \alpha + \beta)x^2 + x(2x + \alpha) + 2$

$= (3 + \alpha + \beta)x^2 + \alpha x + 2$

$\Rightarrow \quad g'(x) = 2(3 + \alpha + \beta)x + \alpha$ and $g''(x) = 2(3 + \alpha + \beta)$

$\therefore \quad g'(1) = 2(3 + \alpha + \beta) + \alpha$ and $g''(2) = 2(3 + \alpha + \beta)$ ...(ii)

From (i) and (ii), we have

$2(3 + \alpha + \beta) + \alpha = \alpha \Rightarrow \alpha + \beta + 3 = 0$ ...(iii)

and $2(3 + \alpha + \beta) = \beta \Rightarrow 2\alpha + \beta + 6 = 0$ ...(iv)

Solving (iii) and (iv), we get $\alpha = -3, \beta = 0$

From (i), $g'(1) = \alpha = -3$ and $g''(2) = \beta = 0$

$\therefore \quad f(x) = x^2 - 3x \Rightarrow f(1) = -2$

and $g(x) = f(1)x^2 + xf'(x) + f''(x)$

$= (-2)(x^2) + x(2x - 3) + 2 = -3x + 2$

**22. (a, b, c, d)**

**23. (b) :** $y = e^{4x}\ e^5 = e^5\ e^{4x}$

$\therefore \quad y_1 = 4^1\ e^5\ e^{4x}, y_2 = 4^2\ e^5\ e^{4x}$, ..... ,

$y_n(x) = 4^n\ e^5\ e^{4x} \Rightarrow y_n(0) = 4^n \cdot e^5 \cdot e^0 = 4^n\ e^5$

**24. (d) :** $y = (2 - 3x)^{-1} \therefore y_1 = (-1)(2 - 3x)^{-2}(-3)$,

$y_2 = (-1)(-2)(2 - 3x)^{-3}(-3)^2$,

$y_3 = (-1)(-2)(-3)(2 - 3x)^{-4}(-3)^3$,

$\vdots \quad \vdots \quad \vdots$

$y_n(x) = (-1)(-2)(-3) \ldots\ldots (-n)(2 - 3x)^{-n-1}(-3)^n$

$\therefore \quad y_n(1) = (-1)^n \cdot n!\ (-3)^n(-1)^{-n-1} = n!(-3)^n(-1)^{-1}$

$= -n!(-3)^n$

**25. (a) :** $y = \dfrac{1}{(x-3)(x-4)} + \sin x + \cos x$

$\Rightarrow \quad y = \left(\dfrac{1}{x-4} - \dfrac{1}{x-3}\right) + \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right)$

$\therefore \quad y_1 = (-1)\left[\dfrac{1}{(x-4)^2} - \dfrac{1}{(x-3)^2}\right] + \sqrt{2}\cos\left(x + \dfrac{\pi}{4}\right)$

$\Rightarrow \quad y_1 = (-1)\left[\dfrac{1}{(x-4)^2} - \dfrac{1}{(x-3)^2}\right] + \sqrt{2}\sin\left(\dfrac{\pi}{2} + \overline{x + \dfrac{\pi}{4}}\right)$

$y_2 = (-1)(-2)\left[\dfrac{1}{(x-4)^3} - \dfrac{1}{(x-3)^3}\right] + \sqrt{2}\sin\left(2 \cdot \dfrac{\pi}{2} + \overline{x + \dfrac{\pi}{4}}\right)$

$\vdots \quad \vdots \quad \vdots$

$y_n = (-1)(-2)(-3)\ldots.(-n)\left[\dfrac{1}{(x-4)^{n+1}} - \dfrac{1}{(x-3)^{n+1}}\right] + \sqrt{2}\sin\left(\dfrac{n\pi}{2} + \overline{x + \dfrac{\pi}{4}}\right)$

$= (-1)^n n!\left[\dfrac{1}{(x-4)^{n+1}} - \dfrac{1}{(x-3)^{n+1}}\right] + \sqrt{2}\sin\left(\dfrac{n\pi}{2} + \overline{x + \dfrac{\pi}{4}}\right)$

**26. P → (1); Q → (3, 4); R → (2, 5)**

**(P)** We have, $3y^2 = ax^2 + b$

$\because$ Point (1, 2) lies on it

$\therefore \quad 3(2)^2 = a(1)^2 + b \Rightarrow a + b = 12$ ...(i)

Also, $6yy' = 2ax$

$\therefore \quad 12 = 2a$ (Given, $\left.\dfrac{dy}{dx}\right|_{(1,2)} = 1$)

$\Rightarrow \quad a = 6 \therefore b = 6$ [Using (i)]

$\therefore \quad a - b = 0$

**(Q)** Given curve is $4y^2 = x^3$

$\because$ Point $(a, b)$ lies on the curve

$\therefore \quad 4b^2 = a^3$ ...(i)

and $8yy' = 3x^2$

$\Rightarrow \quad y' = \dfrac{3x^2}{8y} \quad \therefore \left(\dfrac{dy}{dx}\right)_{(a,b)} = \dfrac{3a^2}{8b}$

Now, as the normal to the curve at $(a, b)$ cuts equal intercept with both axes therefore slope of normal $= \pm 1$.

$\therefore \quad -\left(\dfrac{1}{y'}\right)_{(a,b)} = \pm 1 \Rightarrow \dfrac{3a^2}{8b} = \pm 1 \Rightarrow a^2 = \pm\dfrac{8b}{3}$

$\Rightarrow \quad a^4 = \dfrac{64b^2}{9} = \dfrac{64}{9}\dfrac{a^3}{4}$ [Using (i)]

$\Rightarrow \quad a = 0, \dfrac{16}{9}$, then from (i) $b = 0, \pm\dfrac{32}{27}$

But the line, which makes equal intercept with the coordinate axes cannot pass through (0, 0).

$\therefore \quad$ Required points are $\left(\dfrac{16}{9}, \dfrac{32}{27}\right)$ and $\left(\dfrac{16}{9}, \dfrac{-32}{27}\right)$

$\therefore \quad a - b = \dfrac{16}{9} - \dfrac{32}{27} = \dfrac{16}{27}$ and $a + |b| = \dfrac{16}{9} + \dfrac{32}{27} = \dfrac{80}{27}$

**(R)** Since, (1, 2) lies on $y = ax^2 + bx + \dfrac{7}{2}$

$\therefore \quad 2 = a + b + \dfrac{7}{2} \Rightarrow a + b = -\dfrac{3}{2}$ ...(i)

and $(y')_{(1,2)} = 2(1)a + b = 2a + b$ ...(ii)

and $-\dfrac{1}{\left(\dfrac{dy}{dx}\right)_{(-2,2)}} = -\dfrac{1}{2(-2)+6} = -\dfrac{1}{2}$ ...(iii)

But, $2a + b = -\dfrac{1}{2}$ ...(iv)

Solving (i) and (iv), we get $a = 1$ and $b = -\dfrac{5}{2}$

$\Rightarrow \quad a - b = 1 + \dfrac{5}{2} = \dfrac{7}{2}$ and $5a + 2b = 5 - 5 = 0$

**27.** $\mathbf{P \to (1, 3); Q \to (2, 5); R \to (4)}$

**(P)** For $x \to \infty$, $|x| = x$

$$\therefore \quad f(x)=\left(\frac{x}{x+2}\right)^{-x}=\left(\frac{x+2}{x}\right)^{x}=\left(1+\frac{2}{x}\right)^{x}$$

$$\Rightarrow \lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x}=e^{\lim\limits_{x\to\infty}\left(1+\frac{2}{x}-1\right)\cdot x}=e^{2}$$

and for $x \to -\infty$, $|x| = -x$

$$\therefore \quad f(x)=\left(\frac{-x}{-x+2}\right)^{-x}=\left(\frac{-x+2}{-x}\right)^{x}=\left(1-\frac{2}{x}\right)^{x}$$

$$\Rightarrow \lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\left(1-\frac{2}{x}\right)^{x}$$

$$=e^{\lim\limits_{x\to-\infty}\left(1-\frac{2}{x}-1\right)\cdot x}=e^{-2}$$

$$\textbf{(Q)} \quad f(x)=\frac{(1+x)^{1/x}-e}{x}=\frac{e\left(1-\frac{x}{2}+\frac{11}{24}x^{2}-\ldots\ldots\right)-e}{x}$$

$$\therefore \quad \lim_{x\to0} f(x)=\lim_{x\to0}\frac{e\left(-\frac{x}{2}+\frac{11}{24}x^{2}+\ldots..\right)}{x}=-\frac{e}{2}<-1$$

$$\textbf{(R)} \quad f(x)=\left(\frac{1+7x^{2}}{1+5x^{2}}\right)^{\frac{1}{x^{2}}} \quad \therefore \lim_{x\to0} f(x)=\lim_{x\to0}\left(\frac{1+7x^{2}}{1+5x^{2}}\right)^{\frac{1}{x^{2}}}$$

$$=e^{\lim\limits_{x\to0}\left(\frac{1+7x^{2}}{1+5x^{2}}-1\right)\times\frac{1}{x^{2}}}=e^{\lim\limits_{x\to0}\left(\frac{2}{1+5x^{2}}\right)}=e^{2}$$

**28. (0) :** $f(2x^2 - 1) = (x^3 + x)f(x)$ ...(i)

and $f(2x^2 - 1) = -(x^3 + x)f(-x)$ ...(ii) (Replacing $x$ by $-x$)

From (i) and (ii), we get

$f(-x) = -f(x)$, provided $x \neq 0$ ...(iii)

Now, as $f$ is continuous

$\therefore$ From (iii), we get $f(0) = -f(0) \Rightarrow f(0) = 0$ ...(iv)

$$\text{Now, } \lim_{x\to0}\frac{f(2x^{2}-1)}{x}=\lim_{x\to0}-\frac{(x^{3}+x)f(-x)}{x} \text{ [Using (ii)]}$$

$$=\lim_{x\to0}\frac{x(x^{2}+1)f(x)}{x} \quad \text{[Using (iii)]}$$

$$=\lim_{x\to0}(x^{2}+1)f(x)=\lim_{x\to0}(x^{2}+1)\lim_{x\to0}f(x)$$

$$=1\cdot f(0)=1\times0=0 \quad \text{[Using (iv)]}$$

$$\Rightarrow \lim_{\theta\to0}\frac{f\left(2\sin^{2}\frac{\theta}{2}-1\right)}{\sin\frac{\theta}{2}}=0 \quad \left[\text{By putting } x=\sin\frac{\theta}{2}\right]$$

$$\Rightarrow \lim_{\theta\to0}\frac{f(\cos\theta)}{\sin\frac{\theta}{2}}=0$$

$$\Rightarrow \lim_{\theta\to0}\frac{f(\cos\theta)}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\times2\cos\frac{\theta}{2}=0$$

$$\Rightarrow \lim_{\theta\to0}\frac{f(\cos\theta)}{\sin\theta}=0 \Rightarrow \lim_{x\to0}\frac{f(\cos x)}{\sin x}=0$$

**29. (6) :** $\because \quad f(x)=\int_0^x f(t)dt$

$$\therefore \quad f(x^3)=\int_0^{x^3} f(t)dt=x^3(1+x)$$

$\Rightarrow 3x^2 f(x^3) - f(0)\cdot0 = 3x^2 + 4x^3$

(On differentiating both sides w.r.t $x$)

$\Rightarrow f(x^3)=1+\frac{4}{3}x \quad \therefore \quad f(27) = 5$ (By putting $x = 3$)

$\Rightarrow n = 5$

Now, $(49216)^{n^2} = (49216)^{25}$

$\therefore$ Unit place of $(49216)^{25}$ is 6.

**30. (1) :** $\lim\limits_{x\to0}\frac{x(1+a\cos x)-b\sin x}{(f(x))^{3}}=1$

$$\Rightarrow \lim_{x\to0}\left[\frac{x\left\{1+a\left(1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\ldots\right)\right\}-b\left(x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\ldots\right)}{\left(\frac{f(x)}{x}\right)^{3}\cdot x^{3}}\right]=1$$

$$\Rightarrow \lim_{x\to0}\frac{(1+a-b)x+\left(\frac{b}{3!}-\frac{a}{2!}\right)x^{3}+\left(\frac{a}{4!}-\frac{b}{5!}\right)x^{5}+\ldots}{x^{3}}=1$$

$$\left(\therefore \lim_{x\to0}\frac{f(x)}{x}=1, \text{given}\right)$$

The above limit is possible only if

$1+a-b=0$ and $\frac{b}{3!}-\frac{a}{2!}=1$

(As R.H.S. is a finite quantity with limit = 1)

$\Rightarrow \quad a - b = -1$ ...(i) and $b - 3a = 6$ ...(ii)

Solving (i) and (ii), we get $a=\frac{-5}{2}, \quad b=\frac{-3}{2}$

$$\therefore \quad \frac{a}{b}=\frac{5}{3} \Rightarrow \left[\frac{a}{b}\right]=\left[\frac{5}{3}\right]=1$$

# MOCK TEST PAPER 2020

# JEE MAIN

with Numerical Value Type Questions

**Exam Date : Between 05th, 07th to 09th & 11th April, 2020**

**Time : 1 hr. 15 min.** **Series - 7**

The entire syllabus of Mathematics of JEE MAIN is being divided in to eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given bellow:

| Unit No. | Topic | Syllabus In Details |
|---|---|---|
| UNIT NO.7 | Co-ordinate Geometry-3D | Direction ratios and direction cosines. Angle between two intersecting lines. Skew lines, the shortest distance between them and its equation. Equation of a line and a plane in different forms, intersection of a line and a plane, coplanar lines. |
| | Differential calculus | Rolle's and Lagrange's Mean value theorems, Applications of derivatives: Rate of change of quantities, monotonic-increasing and decreasing functions, Maxima and minima of functions of one variable, tangents and normals. |
| | Integral calculus | Integral as an anti-derivative. Fundamental integrals involving algebraic, trigonometric, exponential and logarithmic functions. Integration by substitution, by parts and by partial fractions. Integration using trigonometric identities. Evaluation of simple integrals of type: $\int\frac{dx}{x^2\pm a^2}, \int\frac{dx}{\sqrt{x^2\pm a^2}}, \int\frac{dx}{a^2-x^2}, \int\frac{dx}{\sqrt{a^2-x^2}}, \int\frac{dx}{ax^2+bx+c}, \int\frac{dx}{\sqrt{ax^2+bx+c}}, \int\frac{(px+q)dx}{ax^2+bx+c}$ |

**1.** If the lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}, \frac{x-1}{3}=\frac{y-2}{-1}=\frac{z-3}{4}$ and $\frac{x+k}{3}=\frac{y-1}{2}=\frac{z-2}{h}$ are concurrent, then

(a) $h=-2, k=-6$ (b) $h=\frac{1}{2}, k=2$

(c) $h=6, k=2$ (d) $h=2, k=\frac{1}{2}$

**2.** The distance between the plane whose equation is $\vec{r}\cdot(2\hat{i}+\hat{j}-3\hat{k})=5$ and the line whose equation is $\vec{r}=\hat{i}+\lambda(2\hat{i}+5\hat{j}+3\hat{k})$, is

(a) $\frac{3}{\sqrt{14}}$ (b) $\frac{5}{\sqrt{14}}$ (c) 5 (d) 0

**3.** Let $\vec{n}$ be a vector of magnitude $2\sqrt{3}$ such that it makes acute equal angles with the co-ordinate axes, the vector form of equation of plane passing through $(1, -1, 2)$ and normal to $\vec{n}$ is

(a) $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=2$ (b) $\vec{r}\cdot(2\hat{i}-3\hat{j}+\hat{k})=2$

(c) $\vec{r}\cdot(2\hat{i}+\hat{j}+\hat{k})=2$ (d) $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=2$

**4.** Let $\vec{A}$ be a vector parallel to the line of intersection of the planes $P_1$ and $P_2$ through the origin. The plane $P_1$ is parallel to the vectors $2\hat{j}+3\hat{k}$ and $4\hat{j}-3\hat{k}$ while the plane $P_2$ is parallel to the vectors $\hat{j}-\hat{k}$ and $3\hat{i}+3\hat{j}$. The angle between $\vec{A}$ and $2\hat{i}+\hat{j}-2\hat{k}$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

**5.** The equation of the plane through the line of intersection of $\vec{r}\cdot(\hat{i}-2\hat{j}+2\hat{k})=1$ and $\vec{r}\cdot(2\hat{i}-\hat{j})+2=0$ and perpendicular to $\vec{r}\cdot(\hat{i}+\hat{j}+2\hat{k})+9=0$ is

(a) $\vec{r}\cdot(-5\hat{i}+\hat{j}+2\hat{k})=7$ (b) $\vec{r}\cdot(5\hat{i}+\hat{j}+2\hat{k})=7$

(c) $\vec{r}\cdot(5\hat{i}-\hat{j}-2\hat{k})=7$ (d) None of these

**6.** The reflection of the plane $2x - 3y + 4z - 3 = 0$ in the plane $x - y + z - 3 = 0$ is the plane

(a) $4x - 3y + 2z - 15 = 0$

(b) $x - 3y + 2z - 15 = 0$

(c) $4x + 3y - 2z + 15 = 0$

(d) None of these

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**7.** $f(x)$ is a polynomial of third degree which has a local maxima at $x = -1$. If $f(1) = -1, f(2) = 18$ and $f'(x)$ has a local minimum at $x = 0$, then

(a) $f(0) = 5$

(b) the local minimum value of $f(x)$ is 1.

(c) $f(x)$ is increasing in $[1, 2\sqrt{5}]$.

(d) The distance between $(-1, 2)$ and $(a, f(a))$ is $2\sqrt{5}$.

**8.** The minimum area of the triangle formed by a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the coordinate axes, is

(a) $ab$ sq. units (b) $\frac{a^2+b^2}{2}$ sq. units

(c) $\frac{(a+b)^2}{2}$ sq. units (d) $\frac{a^2+ab+b^2}{3}$ sq. units

**9.** If $f(x)$ and $g(x)$ are differentiable functions in $[0, 1]$ such that $f(0) = 2, f(1) = 6, g(0) = 0, g(1) = 2$, then there exists $0 < c < 1$ such that

(a) $f'(c) = g'(c)$ (b) $f'(c) = -g'(c)$

(c) $f'(c) = 2g'(c)$ (d) $2f'(c) = g'(c)$

**10.** Electric current $C$, measured by a galvanometer, is given by the equation $C = k\tan\theta$, where $k$ is a constant. Then the percentage error in the current corresponding to an error 0.7 percent in the measurement of $\theta$ when $\theta = 45°$, is

(a) 1.4 (b) 2.8 (c) 1.1 (d) 2.2

**11.** A cone is inscribed in a sphere of radius $r$. The height of the cone when its volume is maximum is

(a) $2r$ (b) $\frac{\sqrt{3}r}{2}$ (c) $\frac{2r}{\sqrt{3}}$ (d) $\frac{4r}{3}$

**12.** The curve $y = ax^3 + bx^2 + cx + 5$ touches the $x$-axis at $P(-2, 0)$ and cut the $y$-axis at a point $Q$ where its gradient is 3. Then $a + b + c =$

(a) $\frac{9}{4}$ (b) $\frac{7}{4}$ (c) $\frac{5}{4}$ (d) $\frac{3}{4}$

**13.** The point on the parabola $2y = x^2$, which is nearest to the point $(0, 3)$ is

(a) $(\pm4, 8)$ (b) $\left(\pm3, \frac{9}{2}\right)$

(c) $(\pm2, 2)$ (d) $\left(\pm1, \frac{1}{2}\right)$

**14.** Three normals are drawn from the point $(c, 0)$ to the curve $y^2 = x$. If two of the normals are perpendicular to each other, then $c =$

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) 1

**15.** In the mean value theorem $f(b) - f(a) = (b - a)f'(c)$ $(a < c < b)$, if $a = 4, b = 9$ and $f(x)=\sqrt{x}$, then the value of $c$ is

(a) 8 (b) 5.25 (c) 4 (d) 6.25

**16.** If $\int\sqrt{\tan x}\,dx = a\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)$ $+b\ln\left(\frac{\tan x+1+\sqrt{2\tan x}}{\tan x+1-\sqrt{2\tan x}}\right)+c$, then $a+b=$

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{2\sqrt{2}}$ (d) $\frac{1}{4}$

**17.** The integral $\int\frac{dx}{\sqrt{x}+\sqrt[3]{x^2}}$ represents the function

(a) $6\left[\sqrt[3]{x^2}-\sqrt[3]{x}+\log\left|1+\sqrt[3]{x}\right|\right]+c$

(b) $3\sqrt[3]{x}-6\sqrt[6]{x}+6\log\left|1+\sqrt[6]{x}\right|+c$

(c) $3\sqrt[3]{x}+6\sqrt[6]{x}+6\log\left|1+\sqrt[6]{x}\right|+c$

(d) $6\sqrt[3]{x^2}-3\sqrt[3]{x}+6\log\left|1+\sqrt[3]{x}\right|+c$

**18.** $\int\frac{e^x dx}{(e^x+2)(e^x+1)}$ is equal to

(a) $\log\left|\frac{e^x+1}{e^x+2}\right|+c$ (b) $\log\left|\frac{e^x+2}{e^x+1}\right|+c$

(c) $\frac{e^x+1}{e^x+2}+c$ (d) $\frac{e^x+2}{e^x+1}+c$

**19.** If $\int\frac{dx}{\cos x+\cos\alpha}=f(\alpha)\log\left|\frac{\cos\frac{x-\alpha}{2}}{\cos\frac{x+\alpha}{2}}\right|+c$,

then the value of $f(\alpha)$ is

(a) $\sin\alpha$ (b) $\cos\alpha$ (c) $\operatorname{cosec}\alpha$ (d) $\sec\alpha$

**20.** If $\int f(x)\sin x\cos x dx=\frac{1}{2(b^2-a^2)}\log(f(x))+c$,

then $\int f(x)dx =$

(a) $\frac{1}{ab}\tan^{-1}\left(\frac{a\tan x}{b}\right)+k$ (b) $ab\tan^{-1}\left(\frac{a\tan x}{b}\right)+k$

(c) $\frac{1}{ab}\tan^{-1}\left(\frac{b\tan x}{a}\right)+k$ (d) $\frac{1}{ab}\tan^{-1}\left(\tan\left(\frac{bx}{a}\right)\right)+k$

## NUMERICAL VALUE TYPE

**21.** The distance of the point (1, –2, 3) from the plane $x - y + z = 5$ measured parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$ is

**22.** If $\int\frac{11\cos x-16\sin x}{2\cos x+5\sin x}dx$ $=-\lambda x+\mu\log|\lambda\cos x+\delta\sin x|+c$, then $(\lambda + \mu) - \delta =$

**23.** If $m$ be the slope of the tangent to the curve $e^y = 1 + x^2$, then $|m| \leq$

**24.** The chord joining the points (5, 5) and (11, 227) on the curve $y = 3x^2 - 11x - 15$ is parallel to tangent at a point on the curve. Then, the abscissa of the point is

**25.** A spherical balloon is being inflated at the rate of 28 $cm^3$/min. Then the rate of increase of surface area of the balloon when its diameter is 14 cm, is

## SOLUTIONS

**1. (d) :** Coordinates of a general point on first line can be taken as $(\lambda, 2\lambda, 3\lambda)$.

Coordinates of a general point on second line can be taken as $(3\mu + 1, -\mu + 2, 4\mu + 3)$.

If the lines intersect, then they have a common point.

So, for some values of $\lambda$ and $\mu$, we must have

$\lambda = 3\mu + 1$, $2\lambda = -\mu + 2$ and $3\lambda = 4\mu + 3$

$\Rightarrow \lambda - 3\mu = 1$, $2\lambda + \mu = 2$, $3\lambda - 4\mu = 3$

Solving two of these equations, we get $\lambda = 1$ and $\mu = 0$.

So, the point of intersection of the first two lines is (1, 2, 3). It lies on the third line. So, $\frac{1+k}{3}=\frac{2-1}{2}=\frac{3-2}{h}$

$\Rightarrow k=\frac{1}{2}$ and $h=2$

**2. (a) :** Here, the vector equations of the plane and the straight line are respectively

$$\vec{r}\cdot(2\hat{i}+\hat{j}-3\hat{k})=5 \quad ...(i)$$

and $\vec{r}=\hat{i}+\lambda(2\hat{i}+5\hat{j}+3\hat{k})$ ...(ii)

Here, $\vec{b}\cdot\vec{n}=(2\hat{i}+5\hat{j}+3\hat{k})\cdot(2\hat{i}+\hat{j}-3\hat{k})=0$

So, $\vec{b}$ is perpendicular to $\vec{n}$.

$\therefore$ Given line is parallel to the given plane.

The distance between the line and the parallel plane is the distance between any point on the line and the given plane. Let the required distance be $d$.

$$\therefore \quad d=\frac{\left|(\hat{i}+0\hat{j}+0\hat{k})\cdot(2\hat{i}+\hat{j}-3\hat{k})-5\right|}{\sqrt{2^2+1^2+3^2}}=\frac{3}{\sqrt{14}}$$

**3. (d) :** Given that $\alpha = \beta = \gamma \Rightarrow \cos\alpha = \cos\beta = \cos\gamma$

$$\therefore \quad l=m=n=\frac{1}{\sqrt{3}} \qquad [\because\ l^2+m^2+n^2=1]$$

$$\therefore \quad \vec{n}=2\sqrt{3}\left(\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right) \qquad [\because \vec{n}=|\vec{r}|(l\hat{i}+m\hat{j}+n\hat{k})]$$

Thus, the required vector equation of plane is

$\vec{r}\cdot(2\hat{i}+2\hat{j}+2\hat{k})=(\hat{i}-\hat{j}+2\hat{k})\cdot(2\hat{i}+2\hat{j}+2\hat{k})$

$\Rightarrow \vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=2$

**4. (b) :** A vector along the line of intersection of two planes one of them being parallel to the vectors $\vec{a}, \vec{b}$ and the other plane being parallel to the vectors $\vec{c}$ and $\vec{d}$, is $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$.

$\therefore \quad \vec{A}=\{(2\hat{j}+3\hat{k})\times(4\hat{j}-3\hat{k})\}\times\{(\hat{j}-\hat{k})\times(3\hat{i}+3\hat{j})\}$

$=-18\hat{i}\times(3\hat{i}-3\hat{j}-3\hat{k})=54(\hat{k}-\hat{j})$

If $\theta$ be an acute angle, then

$$\cos\theta=\left|\frac{54(\hat{k}-\hat{j})}{54\sqrt{2}}\cdot\frac{(2\hat{i}+\hat{j}-2\hat{k})}{\sqrt{9}}\right|=\left|-\frac{1}{\sqrt{2}}\right|=\frac{1}{\sqrt{2}}$$

So, $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

**5. (a) :** The equation of plane through the line of intersection of $\vec{r}\cdot(\hat{i}-2\hat{j}+2\hat{k})=1$ and $\vec{r}\cdot(2\hat{i}-\hat{j})+2=0$ is $[\vec{r}\cdot(\hat{i}-2\hat{j}+2\hat{k})-1]+\lambda[\vec{r}\cdot(2\hat{i}-\hat{j})+2]=0$

$\Rightarrow \vec{r}\cdot[(1+2\lambda)\hat{i}+(-2-\lambda)\hat{j}+2\hat{k}]=1-2\lambda$ ...(i)

Since (i) is perpendicular to $\vec{r}\cdot(\hat{i}+\hat{j}+2\hat{k})+9=0$

$\therefore$ $(1 + 2\lambda)1 + (-2 - \lambda)1 + 2\cdot2 = 0 \Rightarrow \lambda = -3$

Required equation of plane is $\vec{r}\cdot(-5\hat{i}+\hat{j}+2\hat{k})=7$

**6. (a) :** As we know that reflection of $a'x + b'y + c'z + d' = 0$ in the plane $ax + by + cz + d = 0$ is given by

$2(aa' + bb' + cc')(ax + by + cz + d)$

$= (a^2 + b^2 + c^2)(a'x + b'y + c'z + d')$

$\therefore$ $2(2 + 3 + 4)(x - y + z - 3) = 3(2x - 3y + 4z - 3)$

$\Rightarrow$ $6(x - y + z - 3) = 2x - 3y + 4z - 3$

$\Rightarrow$ $4x - 3y + 2z - 15 = 0$ is the required equation of the plane.

**7. (c) :** Let $f(x) = ax^3 + bx^2 + cx + d$

$f(1) = -1 \Rightarrow a + b + c + d = -1$

$f(2) = 18 \Rightarrow 8a + 4b + 2c + d = 18$

$f''(0) = 0 \Rightarrow b = 0$

and $f'(-1) = 0 \Rightarrow 3a - 2b + c = 0$

Solving these equations, we get $a=\frac{19}{4}$, $b=0$,

$c=-\frac{57}{4}$, $d=\frac{17}{2}$ $\therefore$ $f(x)=\frac{1}{4}(19x^3-57x+34)$

$f'(x)=0 \Rightarrow 57x^2-57=0 \Rightarrow x=1,-1$

and $f''(x)=\frac{57}{2}x$

$\because$ $f''(-1)<0$ and $f''(1)>0$.

$\therefore$ $f(x)$ has a local minimum at $x=1$.

$f(x)$ is increasing $\Rightarrow f'(x)\geq 0 \Rightarrow x^2-1\geq 0$

$\Rightarrow$ $x\leq -1$ or $x\geq 1$.

$\therefore$ $f(x)$ is increasing in $[1, 2\sqrt{5}]$.

Also, the distance of $(-1, 2)$ from $(1, -1)$ is not $2\sqrt{5}$.

**8. (a) :** The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a\cos\theta, b\sin\theta)$ is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.

The intercepts of the tangent on the axes are $a\sec\theta$ and $b$ cosec $\theta$.

$\therefore$ Area of the triangle $=\frac{1}{2}\times a\sec\theta\times b\operatorname{cosec}\theta$

$=\frac{ab}{\sin 2\theta}\geq ab$ sq. units

**9. (c) :** Let $F(x)=f(x)-2g(x)$

$\therefore$ $F(0)=f(0)-2g(0)=2$ and $F(1)=f(1)-2g(1)=2$

$F(x)$ satisfies the condition of Rolle's theorem. So, there exists $c$, such that $F'(c)=0$ for $0<c<1$

$\Rightarrow$ $f'(c)-2g'(c)=0$

**10. (c) :** Here, we have

$C=k\tan\theta \Rightarrow \frac{dC}{d\theta}=k\sec^2\theta \Rightarrow \frac{dC}{d\theta}\delta\theta=k\sec^2\theta\delta\theta$

$\Rightarrow$ $\delta C=k\sec^2\theta\delta\theta$ $\left[\because \delta C=\frac{dC}{d\theta}\delta\theta\right]$

Now, $\frac{\delta C}{C}\times 100=\frac{k\sec^2\theta\delta\theta}{k\tan\theta}\times 100=\frac{\sec^2\theta}{\tan\theta}\delta\theta\times 100$

$=2\times\frac{\pi}{4}\frac{\delta\theta}{\theta}\times 100=\frac{\pi}{2}\times 0.7=1.57\times 0.7=1.099$

$\therefore$ Percentage of error in $C$ is 1.1.

**11. (d):** Let $x$ be the radius of the base and $y$ be the height of the cone.

$\therefore$ $x^2=r^2-(y-r)^2=2ry-y^2$

Now, $V=\frac{1}{3}\pi x^2y=\frac{1}{3}\pi(2ry^2-y^3)$

and $\frac{dV}{dy}=0\Rightarrow 4ry=3y^2\Rightarrow y=\frac{4r}{3}$

**12. (b) :** Let $y=f(x)\therefore$ $f(-2)=f'(-2)=0$, $f'(0)=3$, $f(0)=5$.

Now, $f(x)=ax^3+bx^2+cx+5\Rightarrow f'(x)=3ax^2+2bx+c$

and $f'(0)=3\Rightarrow c=3$

and $f'(-2)=0 \Rightarrow 12a-4b+3=0$

$\Rightarrow$ $3a-b=-3/4$ ...(i)

Also, $f(-2)=0 \Rightarrow -8a+4b-2c+5=0$

$\Rightarrow$ $-8a+4b-6+5=0 \Rightarrow -2a+b=1/4$ ...(ii)

Solving (i) and (ii), we get

$a=-\frac{1}{2}$ and $b=-\frac{3}{4}$ $\therefore$ $a+b+c=-\frac{1}{2}-\frac{3}{4}+3=\frac{7}{4}$

**13. (c):** Let the point on the parabola $2y=x^2$ be $\left(x,\frac{x^2}{2}\right)$.

Now, the distance between the point $\left(x,\frac{x^2}{2}\right)$ and $(0, 3)$ be $L$ (say)

So, $L^2=x^2+\left(\frac{x^2}{2}-3\right)^2$

$\frac{d(L^2)}{dx}=2x+2\left(\frac{x^2}{2}-3\right)\times\frac{1}{2}\times 2x=2x\left(1+\frac{x^2}{2}-3\right)$

$=2x\left(\frac{x^2}{2}-2\right)=x^3-4x$ . Also, $\frac{d^2(L^2)}{dx^2}=3x^2-4$

Now, $\frac{d(L^2)}{dx}=0$ gives $x^3-4x=0$

$\Rightarrow$ $x(x^2-4)=0 \Rightarrow x=0$ or $x=\pm 2$

$\therefore$ $\left[\frac{d^2(L^2)}{dx^2}\right]_{x=\pm 2}=8>0$, so $L^2$ is minimum when $x=\pm 2$.

Therefore, the required point is $(\pm 2, 2)$.

**14. (c) :** $x=t^2$, $y=t$, $\frac{dy}{dx}=\frac{1}{2t}$. Therefore, equation of normal is $\frac{1}{2t}(y-t)+(x-t^2)=0$.

The normal passes through $(c,0)\Rightarrow t^2+\frac{1}{2}-c=0$.

$t_1, t_2$ are roots $\Rightarrow$ $t_1t_2=\frac{1}{2}-c$ and slopes $m_1=\frac{1}{2t_1}$,

$m_2=\frac{1}{2t_2}$

$\therefore$ $m_1m_2=-1\Rightarrow\frac{1}{4t_1t_2}=-1 \Rightarrow \frac{1}{2}-c=-\frac{1}{4} \Rightarrow c=\frac{3}{4}$

**15. (d) :** Given function is $f(x)=\sqrt{x}$

$\therefore \quad f'(x)=\dfrac{1}{2\sqrt{x}}$

Now, $\dfrac{f(b)-f(a)}{b-a}=f'(c) \Rightarrow \dfrac{\sqrt{9}-\sqrt{4}}{9-4}=\dfrac{1}{2\sqrt{c}}$

$\Rightarrow \dfrac{2}{5}=\dfrac{1}{\sqrt{c}} \quad \Rightarrow \quad \sqrt{c}=\dfrac{5}{2} \quad \Rightarrow \quad c=6.25$

**16. (c) :** Let $I=\int\sqrt{\tan x}dx$ [Putting $\tan x = t^2$]

$$=2\int\frac{t^2\,dt}{t^4+1}=\int\frac{t^2+1+t^2-1}{t^4+1}dt=\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt+\int\frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt$$

$$=\int\frac{d\left(t-\frac{1}{t}\right)}{\left(t-\frac{1}{t}\right)^2+2}+\int\frac{d\left(t+\frac{1}{t}\right)}{\left(t+\frac{1}{t}\right)^2-2}$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}\ln\left(\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right)+c$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)-\frac{1}{2\sqrt{2}}\ln\left(\frac{\tan x+1+\sqrt{2\tan x}}{\tan x+1-\sqrt{2\tan x}}\right)+c$$

$\therefore \quad a+b=\dfrac{1}{\sqrt{2}}-\dfrac{1}{2\sqrt{2}}=\dfrac{1}{2\sqrt{2}}$

**17. (b) :** Let $I=\int\dfrac{dx}{\sqrt{x}+\sqrt[3]{x^2}}$ [Putting $x=t^6$]

$$=\int\frac{6t^5\,dt}{t^3+t^4}=6\int\frac{t^2\,dt}{t+1}=6\int\left(t-1+\frac{1}{t+1}\right)dt$$

$$=3t^2-6t+6\log|t+1|+c=3\sqrt[3]{x}-6\sqrt[6]{x}+6\log\left|\sqrt[6]{x}+1\right|+c$$

**18. (a) :** Let $I=\int\dfrac{e^x dx}{(e^x+2)(e^x+1)}$

Putting $e^x=z\Rightarrow e^x dx=dz$, we get

$$I=\int\frac{dz}{(z+2)(z+1)}=\int\frac{(z+2)-(z+1)}{(z+2)(z+1)}dz$$

$$=\int\frac{dz}{z+1}-\int\frac{dz}{z+2}=\log|z+1|-\log|z+2|+c$$

$$=\log\left|\frac{z+1}{z+2}\right|+c=\log\left|\frac{e^x+1}{e^x+2}\right|+c$$

**19. (c) :** Let $I=\int\dfrac{dx}{\cos x+\cos\alpha}$

Putting $\tan\dfrac{x}{2}=z\Rightarrow dx=\dfrac{2dz}{1+z^2}$, we get

$$I=\int\frac{2dz}{(z^2+1)\left[\cos\alpha+\frac{1-z^2}{1+z^2}\right]}=\int\frac{2dz}{(1+z^2)\cos\alpha+1-z^2}$$

$$=\int\frac{2dz}{(1+\cos\alpha)-(1-\cos\alpha)z^2}=\frac{2}{(1+\cos\alpha)}\int\frac{dz}{1-\left(z\tan\frac{\alpha}{2}\right)^2}$$

$$=\frac{1}{\cos^2\frac{\alpha}{2}}\cdot\frac{1}{2\tan\frac{\alpha}{2}}\log\left|\frac{1+z\tan\frac{\alpha}{2}}{1-z\tan\frac{\alpha}{2}}\right|+c$$

$$=\operatorname{cosec}\alpha\log\left|\frac{1+\tan\frac{x}{2}\tan\frac{\alpha}{2}}{1-\tan\frac{x}{2}\tan\frac{\alpha}{2}}\right|+c=\operatorname{cosec}\alpha\log\left|\frac{\cos\frac{x-\alpha}{2}}{\cos\frac{x+\alpha}{2}}\right|+c$$

**20. (a) :** Given, $\int f(x)\sin x\cos x dx$

$$=\frac{1}{2(b^2-a^2)}\log f(x)+c$$

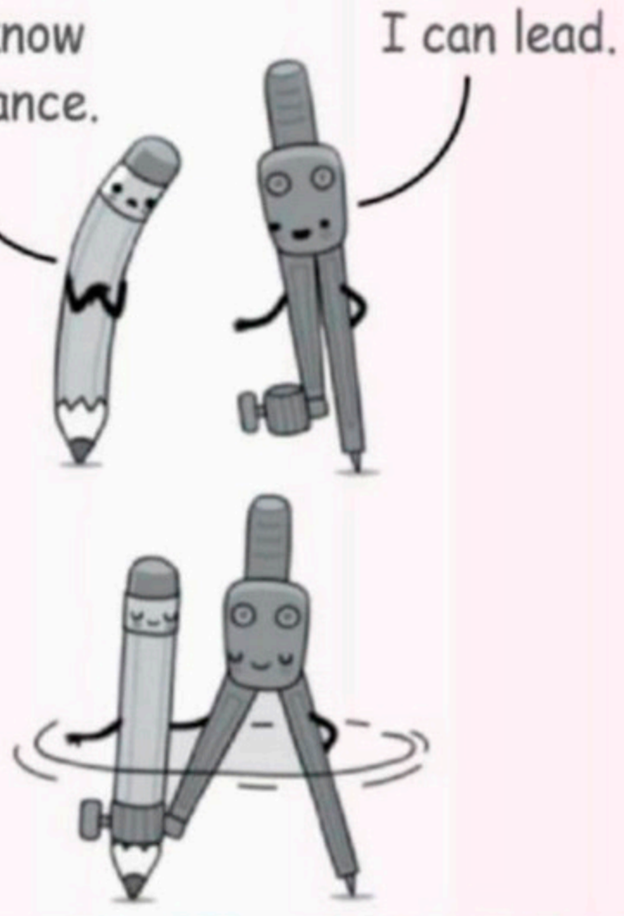
Differentiating both sides w.r.t. $x$, we get

$$f(x)\sin x\cos x=\frac{1}{2(b^2-a^2)}\left[\frac{f'(x)}{f(x)}\right]$$

$$\Rightarrow \quad 2(b^2-a^2)\sin x\cos x=\frac{f'(x)}{\{f(x)\}^2}$$

$$\Rightarrow \int 2(b^2-a^2)\sin x\cos x\,dx=-\frac{1}{f(x)}$$

## COMIC CAPSULE



$$\Rightarrow\ b^2\int 2\sin x\cos x dx - a^2\int 2\sin x\cos x dx = -\frac{1}{f(x)}$$

$$\Rightarrow\ -b^2\cos^2 x - a^2\sin^2 x = -\frac{1}{f(x)}$$

$$\Rightarrow\ f(x) = \frac{1}{b^2\cos^2 x + a^2\sin^2 x} = \frac{\sec^2 x}{b^2 + a^2\tan^2 x}$$

$$\therefore \int f(x)dx = \int \frac{\sec^2 x}{b^2 + a^2\tan^2 x}dx = \frac{1}{ab}\tan^{-1}\left(\frac{a\tan x}{b}\right) + k$$

**21. (1) :** The equation of the line passing through the point (1, –2, 3) and parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$ is

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = \lambda \text{ (say)} \qquad ...(i)$$

$\therefore$ $x = 2\lambda + 1$, $y = 3\lambda - 2$ and $z = -6\lambda + 3$

Now, point of intersection of the line (i) and the plane $x - y + z = 5$ ... (ii) can be obtained by putting $(2\lambda + 1, 3\lambda - 2, -6\lambda + 3)$ in (ii), which gives

$(2\lambda + 1) - (3\lambda - 2) + (-6\lambda + 3) = 5$

$$\Rightarrow\ -7\lambda = -1 \Rightarrow \lambda = \frac{1}{7}$$

$\therefore$ The required point of intersection of the line and plane is $(x, y, z) \equiv \left(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7}\right)$

Thus, the required distance

$$= \sqrt{\left(1 - \frac{9}{7}\right)^2 + \left(-2 + \frac{11}{7}\right)^2 + \left(3 - \frac{15}{7}\right)^2} = \sqrt{\frac{49}{49}} = 1$$

**22. (0) :** $\int \frac{11\cos x - 16\sin x}{2\cos x + 5\sin x}dx$

Let $11\cos x - 16\sin x = l(5\cos x - 2\sin x) + m(2\cos x + 5\sin x)$

Equating the coefficients of $\cos x$ and $\sin x$, we get

$5l + 2m = 11$ ...(i)

and $-2l + 5m = -16 \Rightarrow 2l - 5m = 16$ ...(ii)

Solving (i) and (ii), we get

$l = 3$ and $m = -2$

$$\therefore \int \frac{11\cos x - 16\sin x}{2\cos x + 5\sin x}dx$$

$$= \int \frac{3(5\cos x - 2\sin x) - 2(2\cos x + 5\sin x)}{2\cos x + 5\sin x}dx$$

$$= 3\int \frac{(5\cos x - 2\sin x)}{2\cos x + 5\sin x}dx - 2\int dx$$

$$= 3\log|2\cos x + 5\sin x| - 2x + c$$

$\therefore$ $\lambda = 2$, $\mu = 3$ and $\delta = 5$

So, $(\lambda + \mu) - \delta = (2 + 3) - 5 = 0$

**23. (1) :** Here, the given equation of the curve is $e^y = 1 + x^2$

Differentiating both sides with respect to $x$, we get

$$\frac{dy}{dx} = \frac{2x}{e^y} = \frac{2x}{1 + x^2}$$

$$\therefore\ |m| = \frac{2|x|}{1 + |x|^2} \qquad ...(i)$$

But $(1 - |x|)^2 \geq 0 \Rightarrow 1 + |x|^2 - 2|x| \geq 0 \Rightarrow 1 + |x|^2 \geq 2|x|$

From (i), we get $|m| \leq 1$

**24. (8) :** $y = 3x^2 - 11x - 15 \Rightarrow \frac{dy}{dx} = 6x - 11$

But $6x - 11 = \frac{227 - 5}{11 - 5} = \frac{222}{6} = 37 \Rightarrow 6x = 48 \Rightarrow x = 8$

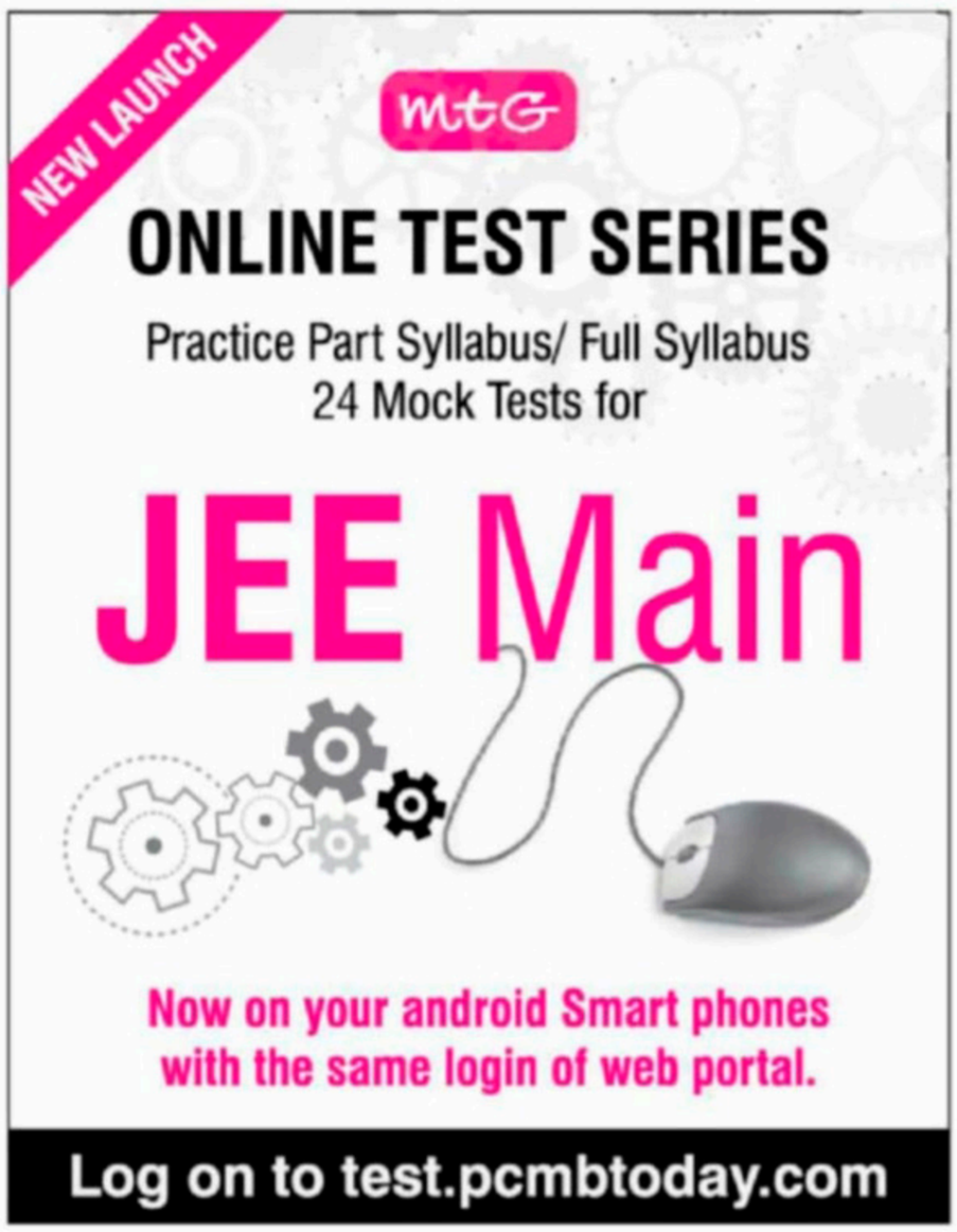
**25. (8) :** Let $s$ and $v$ respectively be the surface area and volume of the spherical balloon.

$$\therefore\ v = \frac{4}{3}\pi r^3 \Rightarrow \frac{dv}{dt} = 4\pi r^2\frac{dr}{dt}$$

Again, $s = 4\pi r^2 \Rightarrow \frac{ds}{dt} = 8\pi r\frac{dr}{dt}$

Now, $\frac{ds/dt}{dv/dt} = \frac{2}{r} \Rightarrow \frac{ds}{dt} = \frac{2}{r}\cdot\frac{dv}{dt} = \frac{2}{7}\times 28 = 8$

Thus, the rate of increase of the surface area is 8 $cm^2$/min, when diameter of the balloon is 14 cm.

# JEE WORK OUTS

JEE Advanced Exam on 17th May, 2020

## One or More Than One Option(s) Correct Type

**1.** The domain of the function

$$f(x)=\log_e\left\{\log_{|\sin x|}(x^2-8x+23)-\frac{3}{\log_2|\sin x|}\right\}$$

contains which of the following interval(s)?

(a) $(3, \pi)$ (b) $\left(\pi, \frac{3\pi}{2}\right)$

(c) $\left(\frac{3\pi}{2}, 5\right)$ (d) none of these

**2.** $\frac{|x^2-4x|+3}{x^2+|x-5|}\geq 1$ is satisfied by which of the following?

(a) $\left(-\infty, -\frac{2}{3}\right]$ (b) $\left[\frac{1}{2}, 2\right]$

(c) $\left(-\infty, \frac{1}{2}\right)$ (d) $\left[-\frac{2}{3}, \infty\right)$

**3.** A particle $P$ starts from the point $z_0 = 1 + 2i$, where $i=\sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point $z_1$. From $z_1$, the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i}+\hat{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anticlockwise direction on a circle with centre at origin, to reach a point $z_2$. The point $z_2$ is given by

(a) $6 + 7i$ (b) $-7 + 6i$ (c) $7 + 6i$ (d) $-6 + 7i$

**4.** Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be arithmetic progressions such that $a_1 = 25$, $b_1 = 75$ and $a_{100} + b_{100} = 100$, then

(a) the difference between successive terms in progression '$a$' is opposite in sign of the difference in progression '$b$'.

(b) $a_n + b_n = 100$ for any $n$.

(c) $(a_1 + b_1), (a_2 + b_2), (a_3 + b_3), \ldots$ are in A.P.

(d) $\sum_{r=1}^{100}(a_r+b_r)=10000$

**5.** If $A$, $B$ are two events such that $P(A \cup B) \geq \frac{3}{4}$ and $\frac{1}{8}\leq P(A\cap B)\leq\frac{3}{8}$, then

(a) $P(A)+P(B)\leq\frac{11}{8}$ (b) $P(A)+P(B)\leq\frac{3}{8}$

(c) $P(A)+P(B)\geq\frac{7}{8}$ (d) none of these

**6.** If $\theta$ is acute angle between the lines $x^2 - 7xy + 12y^2 = 0$, then $\frac{2\cos\theta+3\sin\theta}{4\sin\theta+5\cos\theta}=$

(a) $-39/61$ (b) $61/39$ (c) $29/69$ (d) $69/29$

**7.** If length of latus rectum of ellipse $E_1 : 2(x + y - 1)^2 + (x - y + 3)^2 = 4$ and $E_2 : \frac{x^2}{\lambda}+\frac{y^2}{\lambda^2}=1\,(0<\lambda<1)$ are equal, then

(a) area of ellipse $E_2$ is $\frac{\pi}{\sqrt{2}}$ sq. units

(b) area of director circle of $E_2$ is $\pi\left(2^{-\frac{1}{3}}+2^{-\frac{2}{3}}\right)$ sq. units

(c) area of $E_1$ is $2\pi$ sq. units

(d) none of these

**8.** Let $f(x)=\begin{cases}x^2\left|\cos\frac{\pi}{x}\right|, & x\neq 0\\ 0, & x=0\end{cases}, x\in R$, then $f$ is

(a) differentiable both at $x = 0$ and at $x = 2$
(b) differentiable at $x = 0$ but not differentiable at $x = 2$
(c) not differentiable at $x = 0$ but differentiable at $x = 2$
(d) differentiable at neither at $x = 0$ nor at $x = 2$

**9.** If $\alpha, \beta$ are the roots of equation $x^2 + px + q = 0$ and $\alpha^4, \beta^4$ be those of equation $x^2 - rx + s = 0$ and $f(x) = x^2 - 4qx + 2q^2 - r$, then which one is necessarily true?

(a) $f(p^2) = 0$ (b) $f(q^2) = 0$
(c) $f(r^2) = 0$ (d) $f(s^2) = 0$

**10.** If $(x - a)\cos\theta + y\sin\theta = (x - a)\cos\phi + y\sin\phi = a$ and $\tan(\theta/2) - \tan(\phi/2) = 2b$, then

(a) $y^2 = 2ax - (1 - b^2)x^2$ (b) $\tan\frac{\theta}{2} = \frac{1}{x}(y + bx)$

(c) $y^2 = 2bx - (1 - a^2)x^2$ (d) $\tan\frac{\phi}{2} = \frac{1}{x}(y - bx)$

**11.** Let $f: I \to N$ be defined as $f(n) = \begin{cases} 2n, & \text{if } n > 0 \\ 1 - 2n, & \text{if } n \le 0 \end{cases}$. Then $f^{-1}(n)$ is defined as

(a) $f^{-1}(n) = \begin{cases} -2n, & \text{if } n \text{ is even} \\ 2n - 1, & \text{if } n \text{ is odd} \end{cases}$

(b) $f^{-1}(n) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ \frac{1-n}{2}, & \text{if } n \text{ is odd} \end{cases}$

(c) $f^{-1}(n) = (-1)^n \frac{n}{2}$ (d) $f^{-1}(n) = (-1)^n \frac{n+1}{2}$

**12.** Find the value(s) of $x$ satisfying the equation $\cot^{-1}\frac{x^2 - 1}{2x} + \tan^{-1}\frac{2x}{x^2 - 1} = \frac{2\pi}{3}$.

(a) $\sqrt{3}$ (b) $-1/\sqrt{3}$
(c) $2 - \sqrt{3}$ (d) $-(2 + \sqrt{3})$

**13.** The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is

(a) 0 (b) 2 (c) 1 (d) 3

**14.** If $y = \tan^{-1}\left(\frac{x}{1 \cdot 2 + x^2}\right) + \tan^{-1}\left(\frac{x}{2 \cdot 3 + x^2}\right) + \tan^{-1}\left(\frac{x}{3 \cdot 4 + x^2}\right) + \ldots$ up to $n$ terms. Find $\frac{dy}{dx}$ when $n \to \infty$.

(a) $\frac{1}{1+x^2}$ (b) $\frac{1}{1+x}$ (c) $1 + x^2$ (d) $1 + x$

**15.** The maximum area of the rectangle whose side pass through the vertices of a given rectangle of sides $a$ and $b$ is

(a) $2(ab)$ (b) $\frac{1}{2}(a+b)^2$
(c) $\frac{1}{2}(a^2 + b^2)$ (d) none of these

**16.** $\sum_{m=1}^{n}\left(\sum_{k=1}^{m}\left(\sum_{p=k}^{m} {}^{n}C_m \cdot {}^{m}C_p \cdot {}^{p}C_k\right)\right) =$

(a) $3^n - 2^n$ (b) $4^n - 3^n$
(c) $3^n + 2^n$ (d) $4^n - 1$

**17.** If $L = \lim_{x \to 0} \frac{1}{x^3}\left(\frac{1}{\sqrt{1+x}} - \frac{1+ax}{1+bx}\right)$ exists, then

(a) $a = \frac{1}{4}$ (b) $b = \frac{3}{4}$
(c) $L = -\frac{1}{32}$ (d) $L = \frac{1}{32}$

**18.** Let $g(x)$ be a non-constant twice differentiable function defined such that $y = g(x)$ is symmetric about the line $x = 2$, $I_1 = \int_{-\pi}^{\pi} g(x+2)\sin x\,dx$ and $I_2 = \int_0^4 \frac{1}{1 + e^{g'(x)}}dx$, then

(a) $I_1 = 0$ (b) $I_1 < I_2$ (c) $I_1 > I_2$ (d) $I_2 = 2$

**19.** The area enclosed by circle $x^2 + y^2 = 4$, parabola $y = x^2 + x + 1$, the curve $y = \left[\sin^2\frac{x}{4} + \cos\frac{x}{4}\right]$ and $x$-axis (where $[\cdot]$ is the greatest integer function) is

(a) $\left(\frac{2\pi}{3} - \frac{1}{\sqrt{3}} + \frac{1}{3}\right)$ sq. units
(b) $\left(\frac{2\pi}{3} + \sqrt{3} - \frac{1}{6}\right)$ sq. units
(c) $\left(\frac{\pi}{3} + \frac{1}{\sqrt{3}} - \frac{1}{6}\right)$ sq. units
(d) None of these

**20.** If $\Delta_r = \begin{vmatrix} 2^{r-1} & 2 \cdot 3^{r-1} & 4 \cdot 5^{r-1} \\ x & y & z \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$, then $\sum_{r=1}^{n} \Delta_r$ is equal to

(a) $xyz$ (b) 1 (c) $-1$ (d) 0

## Comprehension Type

**Paragraph for Q. No. 21 to 23**

Let $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$ upto $\infty = A$, then

**21.** $\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\ldots$ upto $\infty$ is equal to

(a) $\frac{1}{2}A$ (b) $A$ (c) $\frac{1}{4}A$ (d) $\frac{3}{4}A$

**22.** $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$ upto $\infty$ is equal to

(a) $\frac{1}{2}A$ (b) $\frac{3}{4}A$ (c) $\frac{1}{4}A$ (d) $A$

**23.** $\frac{3}{4}+\frac{7}{144}+\frac{11}{900}+\ldots$ upto $\infty$ is equal to

(a) $\frac{1}{4}A$ (b) $\frac{3}{4}A$ (c) $\frac{1}{2}A$ (d) $A$

**Paragraph for Q. No. 24 to 26**

Integrals of class functions following a definite pattern can be found by the method of reduction and recursion. Reduction formulas make it possible to reduce an integral dependent on the index $n > 0$, called the order of the integral, to an integral of the same type with a smaller index. Integration by parts helps us to derive reduction formulas.

**24.** If $I_n = \int \frac{dx}{(x^2+a^2)^n}$, then $I_{n+1} + \frac{1-2n}{2n}\cdot\frac{1}{a^2} I_n$ is equal to

(a) $\frac{x}{(x^2+a^2)^n}$ (b) $\frac{1}{2n a^2}\frac{1}{(x^2+a^2)^{n-1}}$

(c) $\frac{1}{2n a^2}\cdot\frac{x}{(x^2+a^2)^n}$ (d) $\frac{1}{2n a^2}\cdot\frac{x}{(x^2+a^2)^{n+1}}$

**25.** If $I_{n,-m} = \int \frac{\sin^n x}{\cos^m x}dx$, then $I_{n,-m} + \frac{n-1}{m-1} I_{n-2,2-m}$ is equal to

(a) $\frac{\sin^{n-1} x}{\cos^{m-1} x}$ (b) $\frac{1}{(m-1)}\frac{\sin^{n-1} x}{\cos^{m-1} x}$

(c) $\frac{1}{(n-1)}\frac{\sin^{n-1} x}{\cos^{m-1} x}$ (d) $\frac{n-1}{m-1}\frac{\sin^{n-1} x}{\cos^{m-1} x}$

**26.** If $u_n = \int \frac{x^n}{\sqrt{ax^2+2bx+c}}dx$, then $(n+1)au_{n+1} + (2n+1)bu_n + nc\,u_{n-1}$ is equal to

(a) $x^{n-1}\sqrt{ax^2+2bx+c}$ (b) $\frac{x^{n-2}}{\sqrt{ax^2+2bx+c}}$

(c) $\frac{x^n}{\sqrt{ax^2+2bx+c}}$ (d) $x^n\sqrt{ax^2+2bx+c}$

## Matrix-Match Type

**27.** Match the following :

| Column-I | | Column-II | |
|---|---|---|---|
| P. | If maximum and minimum values of $\frac{7+6\tan\theta-\tan^2\theta}{(1+\tan^2\theta)}$ for all real values of $\theta \neq \frac{\pi}{2}$ are $\lambda$ and $\mu$ respectively, then | 1. | $\lambda+\mu=2$ |
| Q. | If maximum and minimum values of $5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right)+3$ for all real values of $\theta$ are $\lambda$ and $\mu$ respectively, then | 2. | $\lambda-\mu=6$ |
| R. | If maximum and minimum values of $1+\sin\left(\frac{\pi}{4}+\theta\right)+2\cos\left(\frac{\pi}{4}-\theta\right)$ for all real values of $\theta$ are $\lambda$ and $\mu$ respectively, then | 3. | $\lambda+\mu=6$ |
| | | 4. | $\lambda-\mu=10$ |
| | | 5. | $\lambda-\mu=14$ |

| | P | Q | R |
|---|---|---|---|
| (a) | 3, 4 | 3, 5 | 1, 2 |
| (b) | 1, 4 | 3, 4 | 3, 5 |
| (c) | 2, 3 | 3, 4 | 1, 5 |
| (d) | 3, 4 | 1, 5 | 3, 5 |

**28.** Match the following :

| Column-I | | Column-II | |
|---|---|---|---|
| P. | The distance between the parallel planes $x + 2y - 2z + 1 = 0$ and $2x + 4y - 4z + 5 = 0$ is | 1. | $\frac{7}{\sqrt{3}}$ units |
| Q. | The radius of the circular section of the sphere $\lvert\vec{r}\rvert = 5$ by the plane $\vec{r}\cdot(\hat{i}+\hat{j}-\hat{k}) = 4\sqrt{3}$ is | 2. | 3 units |
| R. | The distance between the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z-2}{1}$ and the plane $x + y + z + 3 = 0$ is | 3. | 13 units |
| S. | The distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and plane $x - y + z = 5$ is | 4. | $\frac{1}{2}$ unit |

| | P | Q | R | S |
|---|---|---|---|---|
| (a) | 1 | 2 | 3 | 4 |
| (b) | 3 | 2 | 4 | 1 |
| (c) | 4 | 2 | 1 | 3 |
| (d) | 4 | 3 | 2 | 1 |

## Numerical Value Type

**29.** If $\frac{1}{\sqrt{4x+1}}\left[\left(\frac{1+\sqrt{4x+1}}{2}\right)^n-\left(\frac{1-\sqrt{4x+1}}{2}\right)^n\right]$ is a fifth degree polynomial, then the sum of the digits of the integer $n$ is

**30.** If the variate takes the values 0, 2, 4, 8, ..., $2^n$ with frequencies ${}^nC_0$, ${}^nC_1$, ..., ${}^nC_n$ respectively and if the mean is $\frac{91\times 8}{2^n}$, then $n$ equals

**31.** From a point, perpendicular tangents are drawn to the ellipse $x^2 + 2y^2 = 2$. The chord of contact touches a circle concentric with the given ellipse. If the sum of the maximum and minimum areas of the circles is $\pi k$. Find $3k$.

**32.** If $\sum_{r=1}^{10}\tan^{-1}\left(\frac{3}{9r^2+3r-1}\right)=\cot^{-1}\left(\frac{m}{n}\right)$ (where $m$ and $n$ are coprime), then find $(2m + n)$.

**33.** Let $f:[1,\infty)\to[2,\infty)$ be a differentiable function such that $f(1)=\frac{1}{3}$. If $6\int_1^x f(t)dt=3xf(x)-x^3$ for all $x \geq 1$, then the value of $3f(2)$ is

**34.** Suppose that $\vec{p}$, $\vec{q}$ and $\vec{r}$ are three non-coplanar vectors $\vec{s}$ in $R^3$. Let the components of a vector $\vec{s}$ along $(-\vec{p}+\vec{q}+\vec{r})$, be 4, 3 and 5 respectively. If the components of this vector $\vec{s}$ along $(-\vec{p}+\vec{q}+\vec{r})$, $(\vec{p}-\vec{q}+\vec{r})$ and $(-\vec{p}-\vec{q}+\vec{r})$ are $x$, $y$ and $z$ respectively, then the value of $2x + y + z$ is

**35.** If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ is $\sqrt{6}$, then $|d|$ is

**36.** Of the three independent events, $E_1$, $E_2$ and $E_3$ the probability that only $E_1$ occurs is $\alpha$, only $E_2$ occurs is $\beta$ and only $E_3$ occurs is $\gamma$. Let the probability $p$ that none of the events $E_1$, $E_2$ or $E_3$ occurs satisfy the equations $(\alpha - 2\beta)p = \alpha\beta$ and $(\beta - 3\gamma)p = 2\beta\gamma$. All the given probabilities are assumed to lie in the interval (0, 1).

Then, $\frac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3} =$

## SOLUTIONS

**1. (a, b, c) :** $f(x)$ is defined if

$$\log_{|\sin x|}(x^2-8x+23)-\frac{3}{\log_2|\sin x|}>0$$

or $\log_{|\sin x|}\left(\frac{x^2-8x+23}{8}\right)>0$

This is true if $|\sin x| \neq 0, 1$ and $\frac{x^2-8x+23}{8}<1$.

Now, $\frac{x^2-8x+23}{8}<1$ or $x^2 - 8x + 15 < 0$

or $x\in(3,5)-\left\{\pi,\frac{3\pi}{2}\right\}$

$$\text{Domain}=(3,\pi)\cup\left(\pi,\frac{3\pi}{2}\right)\cup\left(\frac{3\pi}{2},5\right)$$

**2. (a, b) :** $\frac{|x^2-4x|+3}{x^2+|x-5|}\geq 1$

$\Rightarrow$ $|x^2 - 4x| + 3 \geq x^2 + |x - 5|$ ($\because$ $x^2 + |x - 5| > 0$)

$\Rightarrow$ $|x(x - 4)| - |x - 5| + 3 - x^2 \geq 0$

**Case I :** $x \in (-\infty, 0]$,

$x^2 - 4x - (5 - x) + 3 - x^2 \geq 0 \Rightarrow 3x + 2 \leq 0$

$\Rightarrow$ $x\leq-\frac{2}{3}$ $\therefore$ $x\in\left(-\infty,-\frac{2}{3}\right]$

**Case II :** $x \in (0, 4)$, $-x^2 + 4x - (5 - x) + 3 - x^2 \geq 0$

$\Rightarrow$ $2x^2 - 5x + 2 \leq 0 \Rightarrow (2x - 1)(x - 2) \leq 0$

$\Rightarrow$ $x\in\left[\frac{1}{2},2\right]$

**Case III :** $x \in [4, 5]$, $x^2 - 4x - (5 - x) + 3 - x^2 \geq 0$

$\Rightarrow -3x - 2 \geq 0$

$\Rightarrow$ $x \leq -2/3$, which is not possible.

**Case IV :** $x \in (5, \infty)$, $x^2 - 4x - (x - 5) + 3 - x^2 \geq 0$

$\Rightarrow -5x + 8 \geq 0$

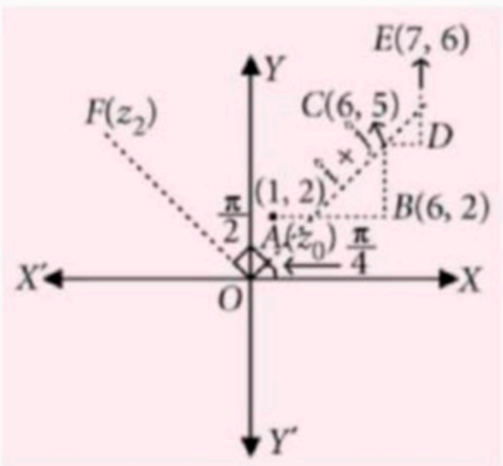
$\Rightarrow$ $5x \leq 8 \Rightarrow x\leq\frac{8}{5}$, which is not possible.

Hence, $x\in\left(-\infty,-\frac{2}{3}\right]\cup\left[\frac{1}{2},2\right]$

**3. (d) :** It is given that $CE$ is in the direction of vector $\hat{i}+\hat{j}$ which makes 45° with $X$-axis. Also, $CE = \sqrt{2}$.

$\therefore$ $CD = DE = 1$ unit.

So, the coordinates of $E$ are (7, 6). $OE$ is rotated through an angle $\pi/2$ in anticlockwise direction to reach a point $F(z_2)$.




$$z_2=\overline{OE}\,e^{i\pi/2}=(7+6i)i=-6+7i$$

**4. (a, b, c, d) :** Here $a_1$, $(a_1 + d)$, $(a_1 + 2d)$, ... and $b_1$, $(b_1 + d_1)$, $(b_1 + 2d_1)$,.... are in A.P.

$\therefore \quad a_{100} = a_1 + 99d$ ...(i) $\quad b_{100} = b_1 + 99d_1$ ...(ii)

Adding (i) and (ii), we get $a_{100} + b_{100} = 100 + 99\,(d + d_1)$

Hence, $d + d_1 = 0 \Rightarrow d = -d_1 \Rightarrow$ options (a), (b) and (c) are obviously true.

(d) $\sum_{r=1}^{100}(a_r + b_r) = \frac{100}{2}[(a_1 + b_1) + (a_{100} + b_{100})]$

$= \frac{100 \times 200}{2} = 10^4$

**5. (a, c) :** $P(A \cup B) \geq \frac{3}{4}$ ...(i) $\quad \frac{1}{8} \leq P(A \cap B) \leq \frac{3}{8}$ ...(ii)

Let $P(A) + P(B)$ be $x$

From (i), $x - P(A \cap B) \geq \frac{3}{4} \Rightarrow x - \frac{3}{4} \geq P(A \cap B) \geq \frac{1}{8}$ [From (ii)]

$\Rightarrow \quad x \geq \frac{7}{8}$

$\because \quad P(A \cup B) \leq 1 \Rightarrow x - P(A \cap B) \leq 1$

$\Rightarrow \quad x - 1 \leq P(A \cap B) \leq \frac{3}{8} \Rightarrow x \leq \frac{11}{8}$ [From (ii)]

**6. (c)**

**7. (a, b) :** $2(x + y - 1)^2 + (x - y + 3)^2 = 4$

$\Rightarrow \quad \frac{\left(\frac{x+y-1}{\sqrt{2}}\right)^2}{1^2} + \frac{\left(\frac{x-y+3}{\sqrt{2}}\right)^2}{(\sqrt{2})^2} = 1$

Latus rectum $= \frac{2a^2}{b} = \sqrt{2}$ ...(i)

From $\frac{x^2}{\lambda} + \frac{y^2}{\lambda^2} = 1$,

Latus rectum $= \frac{2\lambda^2}{\sqrt{\lambda}} = \sqrt{2}$ [From (i)]

$\therefore \quad \lambda^{3/2} = \frac{1}{\sqrt{2}} \Rightarrow \lambda = 2^{-\frac{1}{3}}$

Area of ellipse $E_2$ is $\pi\lambda^{3/2} = \frac{\pi}{\sqrt{2}}$ sq. units

Area of director circle of $E_2 = \pi(\lambda^2 + \lambda)$

$= \pi\left(2^{-\frac{2}{3}} + 2^{-\frac{1}{3}}\right)$ sq. units

Area of $E_1 = \pi\sqrt{2}$ sq. units

**8. (b)**

**9. (a) :** $\alpha + \beta = -p$, $\alpha\beta = q$, $\alpha^4 + \beta^4 = r$, $\alpha^4\beta^4 = s$

Now, $(\alpha + \beta)^4 = \alpha^4 + 4\alpha^3\beta + 6\alpha^2\beta^2 + 4\alpha\beta^3 + \beta^4$

$\Rightarrow \quad p^4 = \alpha^4 + \beta^4 + 4\alpha\beta(\alpha^2 + \beta^2) + 6\alpha^2\beta^2$

$\Rightarrow \quad p^4 = r + 4q(p^2 - 2q) + 6q^2$

$\Rightarrow \quad p^4 - 4p^2q + 2q^2 - r = 0 \Rightarrow f(p^2) = 0$

**10. (a, b, d) :** Let $\tan(\theta/2) = \alpha$ and $\tan(\phi/2) = \beta$

$\therefore \quad \alpha - \beta = 2b$

Also, $\cos\theta = \frac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)} = \frac{1 - \alpha^2}{1 + \alpha^2}$ and

$\sin\theta = \frac{2\tan(\theta/2)}{1 + \tan^2(\theta/2)} = \frac{2\alpha}{1 + \alpha^2}$

Similarly, $\cos\phi = \frac{1 - \beta^2}{1 + \beta^2}$ and $\sin\phi = \frac{2\beta}{1 + \beta^2}$

Therefore, from the given relations, we have

$(x - a)\left(\frac{1 - \alpha^2}{1 + \alpha^2}\right) + y\left(\frac{2\alpha}{1 + \alpha^2}\right) = a$

$\Rightarrow \quad x\alpha^2 - 2y\alpha + 2a - x = 0$

Similarly, $x\beta^2 - 2y\beta + 2a - x = 0$.

We see that $\alpha$ and $\beta$ are the roots of the equation $xz^2 - 2yz + 2a - x = 0$, so that $\alpha + \beta = 2y/x$ and $\alpha\beta = (2a - x)/x$.

Now, from $(\alpha + \beta)^2 = (\alpha - \beta)^2 + 4\alpha\beta$, we get

$\left(\frac{2y}{x}\right)^2 = (2b)^2 + \frac{4(2a - x)}{x} \Rightarrow y^2 = 2ax - (1 - b^2)x^2$

Also, from $\alpha + \beta = 2y/x$ and $\alpha - \beta = 2b$, we get

$\alpha = y/x + b$ and $\beta = y/x - b$

$\Rightarrow \tan\frac{\theta}{2} = \frac{1}{x}(y + bx)$ and $\tan\frac{\phi}{2} = \frac{1}{x}(y - bx)$

**11. (b)**

**12. (a, b, c, d) : Case I :** when $\frac{x^2 - 1}{2x} > 0$

$\cot^{-1}\frac{x^2 - 1}{2x} + \tan^{-1}\frac{2x}{x^2 - 1} = \frac{2\pi}{3}$

$\Rightarrow \quad \tan^{-1}\frac{2x}{x^2 - 1} + \tan^{-1}\frac{2x}{x^2 - 1} = \frac{2\pi}{3}$

$\Rightarrow \quad \tan^{-1}\frac{2x}{x^2 - 1} = \frac{\pi}{3} \quad \Rightarrow \quad \frac{2x}{x^2 - 1} = \sqrt{3}$

$\Rightarrow \quad \sqrt{3}x^2 - 2x - \sqrt{3} = 0 \quad \Rightarrow \quad x = \frac{-1}{\sqrt{3}}, \sqrt{3}$

**Case II :** when $\frac{x^2 - 1}{2x} < 0$

$\cot^{-1}\frac{x^2 - 1}{2x} + \tan^{-1}\frac{2x}{x^2 - 1} = \frac{2\pi}{3}$

$\Rightarrow \quad \pi + \tan^{-1}\frac{2x}{x^2 - 1} + \tan^{-1}\frac{2x}{x^2 - 1} = \frac{2\pi}{3}$

$\left(\because \quad \cot^{-1}x = \pi + \tan^{-1}\frac{1}{x}; x < 0\right)$

$\Rightarrow \quad \tan^{-1}\frac{2x}{x^2 - 1} = \frac{-\pi}{6} \Rightarrow \frac{2x}{x^2 - 1} = \frac{-1}{\sqrt{3}}$

$\Rightarrow \quad x^2 + 2\sqrt{3}x - 1 = 0 \quad \Rightarrow \quad x = -\sqrt{3} \pm 2$

**13. (c)**

**14. (a) :** $y=\tan^{-1}\left(\frac{x}{1\cdot2+x^2}\right)+\tan^{-1}\left(\frac{x}{2\cdot3+x^2}\right)$

$$+\tan^{-1}\left(\frac{x}{3\cdot4+x^2}\right)+\ldots+\tan^{-1}\left(\frac{x}{n(n+1)+x^2}\right)$$

$$\therefore \quad T_r=\tan^{-1}\left(\frac{x}{r(r+1)+x^2}\right)=\tan^{-1}\left(\frac{\frac{x}{r(r+1)}}{1+\frac{x^2}{r(r+1)}}\right)$$

$$\Rightarrow T_r=\tan^{-1}\left(\frac{\frac{x}{r}-\frac{x}{r+1}}{1+\frac{x}{r}\cdot\frac{x}{r+1}}\right)=\tan^{-1}\left(\frac{x}{r}\right)-\tan^{-1}\left(\frac{x}{r+1}\right)$$

$$\therefore \quad y=S_n=\sum_{r=1}^{n}T_r=\sum_{r=1}^{n}\left[\tan^{-1}\left(\frac{x}{r}\right)-\tan^{-1}\left(\frac{x}{r+1}\right)\right]$$

$$=\tan^{-1}(x)-\tan^{-1}\left(\frac{x}{n+1}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{1+x^2}-\frac{1}{1+\left(\frac{x}{n+1}\right)^2}\cdot\left(\frac{1}{n+1}\right)$$

$$\therefore \quad \lim_{n\to\infty}\left(\frac{dy}{dx}\right)=\frac{1}{1+x^2}-0=\frac{1}{1+x^2}$$

**15. (b) :** Area of required rectangle,

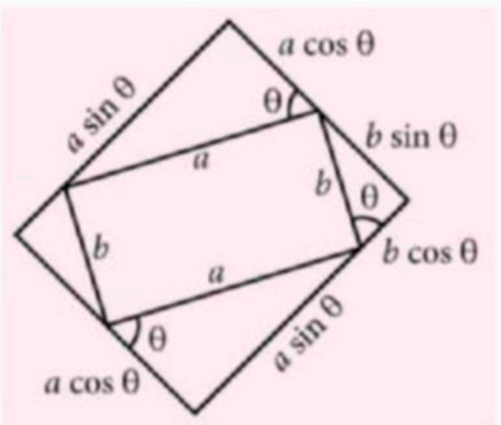
$A = (a \sin\theta + b\cos\theta)$

$(a\cos\theta + b\sin\theta)$

$= ab(\sin^2\theta + \cos^2\theta)$

$+ (a^2 + b^2)\sin\theta\cos\theta$

$$=ab+\frac{(a^2+b^2)}{2}\sin2\theta$$




$A$ is maximum when $\sin 2\theta$ is maximum.

Therefore, $A_{max}=ab+\frac{(a^2+b^2)}{2}=\frac{1}{2}(a+b)^2$

**16. (b)**

**17. (a, b, c) :** $L=\lim_{x\to0}\frac{1}{x^3}\left(\frac{1}{\sqrt{1+x}}-\frac{1+ax}{1+bx}\right)$

$$=\lim_{x\to0}\frac{1+bx-(1+ax)\sqrt{1+x}}{x^3}\lim_{x\to0}\frac{1}{\sqrt{1+x}(1+bx)}$$

$$=\lim_{x\to0}\frac{1+bx-(1+ax)(1+x)^{1/2}}{x^3}$$

$$=\lim_{x\to0}\frac{1+bx-(1+ax)\left(1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3-\ldots\right)}{x^3}$$

$$=\lim_{x\to0}\frac{1+bx-(1+ax)\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\ldots\right)}{x^3}$$

$$=\lim_{x\to0}\frac{\left(b-\frac{1}{2}-a\right)x+\left(-\frac{a}{2}+\frac{1}{8}\right)x^2+\left(\frac{a}{8}-\frac{1}{16}\right)x^3+\ldots}{x^3}$$

Limit exists, if $b-\frac{1}{2}-a=0$ and $\frac{1}{8}-\frac{a}{2}=0$

$$\therefore \quad a=\frac{1}{4} \text{ and } b=\frac{3}{4} \quad \therefore \quad L=-\frac{1}{32}$$

**18. (a, b, d) :** Here, $g(2-x)=g(2+x)$

Hence, $g(2+x)\sin x$ is an odd function.

$\therefore \quad I_1=0$

Now, $g(2-(2-x))=g(2+2-x) \Rightarrow g(x)=g(4-x)$

$\therefore \quad g'(x)=-g'(4-x)$

Also, $I_2=\int_0^4\frac{1}{1+e^{g'(x)}}dx$ ...(i) $\Rightarrow I_2=\int_0^4\frac{1}{1+e^{g'(4-x)}}dx$

$$\Rightarrow I_2=\int_0^4\frac{1}{1+e^{-g'(x)}}dx \quad \Rightarrow \quad I_2=\int_0^4\frac{e^{g'(x)}}{e^{g'(x)}+1}dx \quad ...(ii)$$

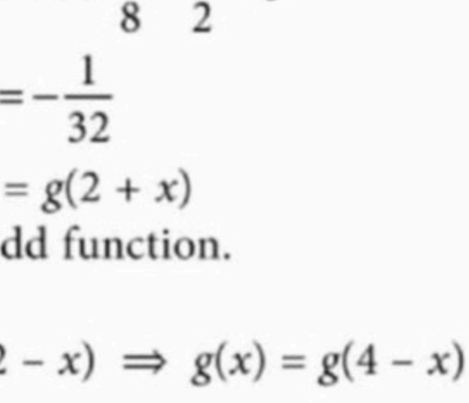
Adding (i) and (ii), we get

$$2I_2=\int_0^4 1dx \quad \therefore \quad I_2=2$$

**19. (b) :** For $x\in(-2,2)$,

$$y=\left[\sin^2\frac{x}{4}+\cos\frac{x}{4}\right]=1$$

Now, we have to find out the area enclosed by the circle $x^2+y^2=4$, parabola $\left(y-\frac{3}{4}\right)=\left(x+\frac{1}{2}\right)^2$, line $y=1$ and $x$-axis.




Hence, required area of the shaded portion

$$=\sqrt{3}\times1+(\sqrt{3}-1)\times1+\int_{-1}^{0}(x^2+x+1)dx$$

$$+2\int_{\sqrt{3}}^{2}(\sqrt{4-x^2})dx$$

$$= (2\sqrt{3}-1)+\left[\frac{x^3}{3}+\frac{x^2}{2}+x\right]_{-1}^{0} +2\left[\frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

$$= (2\sqrt{3}-1)+\left[0-\left(-\frac{1}{3}+\frac{1}{2}-1\right)\right]+2\left[(0+\pi)-\left(\frac{\sqrt{3}}{2}+\frac{2\pi}{3}\right)\right]$$

$$= (2\sqrt{3}-1)+\frac{5}{6}+\frac{2\pi}{3}-\sqrt{3} = \left(\frac{2\pi}{3}+\sqrt{3}-\frac{1}{6}\right)\text{sq. units}$$

**20. (d) :** Using the properties of determinants, we have

$$\sum_{r=1}^{n}\Delta_r = \begin{vmatrix} \sum_{r=1}^{n}2^{r-1} & \sum_{r=1}^{n}2\cdot 3^{r-1} & \sum_{r=1}^{n}4\cdot 5^{r-1} \\ x & y & z \\ 2^n-1 & 3^n-1 & 5^n-1 \end{vmatrix}$$

Now, $\sum_{r=1}^{n}2^{r-1} = 1+2+2^2+\ldots+2^{n-1} = 1.\frac{(2^n-1)}{(2-1)} = 2^n-1,$

$\sum_{r=1}^{n}2\cdot 3^{r-1} = 2[1+3+3^2+\ldots+3^{n-1}] = 3^n-1$

and $\sum_{r=1}^{n}4\cdot 5^{r-1} = 4[1+5+5^2+\ldots+5^{n-1}] = 5^n-1$

$$\therefore \sum_{r=1}^{n}\Delta_r = \begin{vmatrix} 2^n-1 & 3^n-1 & 5^n-1 \\ x & y & z \\ 2^n-1 & 3^n-1 & 5^n-1 \end{vmatrix} = 0$$

[$\because$ $R_1$ and $R_3$ are identical]

**21. (c)** **22. (b)**

**23. (c) :** $\frac{3}{4}+\frac{7}{144}+\frac{11}{900}+\ldots$ up to $\infty$

$$= \frac{3}{1^2\times 2^2}+\frac{7}{3^2\times 4^2}+\frac{11}{5^2\times 6^2}+\ldots \text{ up to } \infty$$

$$= \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\ldots. \text{ up to } \infty$$

$$= \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\ldots \text{ up to } \infty\right) -2\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots \text{upto } \infty\right)$$

$$= A-2\left(\frac{A}{4}\right) = \frac{A}{2}$$

**24. (c) :** Using integration by parts, we get

$$I_n = \frac{x}{(x^2+a^2)^n}+2n\int\frac{x^2}{(x^2+a^2)^{n+1}}dx$$

$$= \frac{x}{(x^2+a^2)^n}+2n\int\frac{1}{(x^2+a^2)^n}dx-2na^2\int\frac{1}{(x^2+a^2)^{n+1}}dx$$

Hence, $I_{n+1}+\frac{1-2n}{2n}\frac{1}{a^2}I_n = \frac{1}{2n\,a^2}\cdot\frac{x}{(x^2+a^2)^n}$

**25. (b) :** $I_{n,-m} = \frac{\sin^{n-1}x}{(m-1)\cos^{m-1}x}-\frac{n-1}{m-1}\int\frac{\sin^{n-2}x}{\cos^{m-2}x}dx$

$$= \frac{\sin^{n-1}x}{(m-1)\cos^{m-1}x}-\frac{n-1}{m-1}I_{n-2,\,2-m}$$

**26. (d) :** Consider $u_{n+1} = \int\frac{x^{n+1}}{\sqrt{ax^2+2bx+c}}dx$

$$= \frac{1}{2a}\int\frac{x^n(2ax+2b)-2bx^n}{\sqrt{ax^2+2bx+c}}dx$$

$$= \frac{1}{2a}\int\frac{x^n(2ax+2b)}{\sqrt{ax^2+2bx+c}}dx-\frac{b}{a}u_n = I_n-\frac{b}{a}u_n, \qquad \ldots(\text{i})$$

where $I_n = \frac{1}{2a}\int\frac{x^n(2ax+2b)}{\sqrt{ax^2+2bx+c}}dx$

$$= \frac{1}{2a}\left[x^n 2\sqrt{ax^2+bx+c}-\int nx^{n-1}2\sqrt{ax^2+2bx+c}\,dx\right]$$

$$= \frac{x^n}{a}\sqrt{ax^2+2bx+c}-\frac{n}{a}\int\frac{x^{n-1}(ax^2+2bx+c)}{\sqrt{ax^2+2bx+c}}dx$$

$\Rightarrow aI_n = x^n\sqrt{ax^2+2bx+c}-na\,u_{n+1}-2bn\,u_n-nc\,u_{n-1}$

Putting this value in (i), we have

$a\,u_{n+1} = x^n\sqrt{ax^2+2bx+c}-na\,u_{n+1}-2bn\,u_n - nc\,u_{n-1}-b\,u_n$

$\Rightarrow (n+1)\,au_{n+1}+(2n+1)\,bu_n+ncu_{n-1} = x^n\sqrt{ax^2+2bx+c}$

**27. (a)**

**28. (c) :** **(P)** Distance between $x+2y-2z+1=0$ and $x+2y-2z+5/2=0$ is $\frac{|5/2-1|}{\sqrt{1+4+4}} = \frac{1}{2}$ unit

**(Q)** Radius of the sphere = 5

Given plane is $x+y-z = 4\sqrt{3}$

Length of the $\perp$ from the centre (0, 0, 0) of the sphere on the plane $= \frac{4\sqrt{3}}{\sqrt{1+1+1}} = 4$

Hence, the radius of the circular section

$= \sqrt{25-16} = \sqrt{9} = 3$ units

**(R)** Line is along the vector $\vec{a} = -3\hat{i}+2\hat{j}+\hat{k}$ and plane is normal to the vector $\vec{b} = \hat{i}+\hat{j}+\hat{k}$

Since $\vec{a}\cdot\vec{b} = 0$, so, line is parallel to the plane.

Hence distance between line and plane is distance of point $(-1, 3, 2)$ from the plane which is $\frac{|-1+3+2+3|}{\sqrt{1+1+1}} = \frac{7}{\sqrt{3}}$ units

**(S)** Any point on the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = \lambda$ is $(3\lambda+2,\ 4\lambda-1,\ 12\lambda+2)$

This point lies on plane $x - y + z = 5$ if $3\lambda + 2 - 4\lambda + 1 + 12\lambda + 2 = 5 \Rightarrow \lambda = 0$

So, point is $(2, -1, 2)$. Its distance from $(-1, -5, -10)$

$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2} = 13$ units

**29. (2) :** Let $\sqrt{4x+1} = t$.

Now, $\frac{1}{t}\left[\left(\frac{1+t}{2}\right)^n - \left(\frac{1-t}{2}\right)^n\right] = \frac{1}{2^n}\left[2\cdot\binom{n}{1} + 2\cdot\binom{n}{3}t^2 + ..\right]$

$= \frac{1}{2^{n-1}}\left[\binom{n}{1} + \binom{n}{3}(4x+1) + \binom{n}{5}(4x+1)^2 + \binom{n}{7}(4x+1)^3 + \binom{n}{9}(4x+1)^4 + \binom{n}{11}(4x+1)^5\right]$

is a fifth degree polynomial if $n = 11$.

**30. (6) :** Mean $= \frac{0\cdot {}^nC_0 + 2\cdot {}^nC_1 + 4\cdot {}^nC_2 + ... + 2^n\cdot {}^nC_n}{{}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n} = \frac{91\times 8}{2^n}$

$\Rightarrow \quad \frac{3^n - 1}{2^n} = \frac{91\times 8}{2^n} \Rightarrow 3^n - 1 = 728 \Rightarrow n = 6$

**31. (5) :** Ellipse : $\frac{x^2}{2} + \frac{y^2}{1} = 1$

Equation of director circle is $x^2 + y^2 = 3$

Any point on director circle is $P(\sqrt{3}\cos\theta, \sqrt{3}\sin\theta)$

Equation of chord of contact of ellipse w.r.t. point $P$ is

$(\sqrt{3}\cos\theta)x + (2\sqrt{3}\sin\theta)y - 2 = 0$

Let it touches circle $x^2 + y^2 = r^2$

$\therefore \quad r = \left|\frac{-2}{\sqrt{3\cos^2\theta + 12\sin^2\theta}}\right| \Rightarrow \quad r^2 = \frac{4}{3 + 9\sin^2\theta}$

$\therefore \quad r^2_{max} = \frac{4}{3+0} = \frac{4}{3}$ and $r^2_{min} = \frac{4}{3+9} = \frac{4}{12} = \frac{1}{3}$

$\therefore$ Max. area of circle + Min. area of circle

$= \pi\left(\frac{4}{3}\right) + \pi\left(\frac{1}{3}\right) = \frac{5}{3}\pi$. So, $k = \frac{5}{3} \Rightarrow 3k = 5$

**32. (32)**

**33. (8) :** We have, $6\int_1^x f(t)dt = 3xf(x) - x^3$

On differentiating, we get

$6f(x) = 3xf'(x) + 3f(x) - 3x^2 \Rightarrow f(x) = xf'(x) - x^2$

$\Rightarrow y = x\frac{dy}{dx} - x^2 \Rightarrow \frac{dy}{dx} - \frac{1}{x}y = x$

I.F. $= e^{\int \frac{-1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

General solution is $y.\frac{1}{x} = \int x.\frac{1}{x}dx + k = x + k$

$\Rightarrow \quad y = x^2 + kx$

At $x = 1$ and $y = 1/3 \quad \Rightarrow \quad k = -2/3$

Thus, $y(x) = x^2 + \frac{-2}{3}x, \quad \therefore \quad y(2) = 4 - \frac{4}{3} = \frac{8}{3}$

**34. (9) :** $\vec{s} = 4\vec{p} + 3\vec{q} + 5\vec{r}$

Let $-\vec{p} + \vec{q} + \vec{r} = \vec{\alpha}$, $\vec{p} - \vec{q} + \vec{r} = \vec{\beta}$, $-\vec{p} - \vec{q} + \vec{r} = \vec{\gamma}$

Then, solving for $\vec{p}$, $\vec{q}$ and $\vec{r}$, we have

$\vec{p} = \frac{\vec{\beta} - \vec{\gamma}}{2}$, $\vec{q} = \frac{\vec{\alpha} - \vec{\gamma}}{2}$, $\vec{r} = \frac{\vec{\alpha} + \vec{\beta}}{2}$

Rewrite, $\vec{s} = 4\left(\frac{\vec{\beta} - \vec{\gamma}}{2}\right) + 3\left(\frac{\vec{\alpha} - \vec{\gamma}}{2}\right) + 5\left(\frac{\vec{\alpha} + \vec{\beta}}{2}\right)$

$= 4\vec{\alpha} + \frac{9}{2}\vec{\beta} - \frac{7}{2}\vec{\gamma} \quad \therefore \quad x = 4, \ y = \frac{9}{2}, \ z = \frac{-7}{2}$

Thus, $2x + y + z = 9$

**35. (6) :** Equation of plane is $\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

$\Rightarrow \quad x - 2y + z = 0$

Comparing with $Ax - 2y + z = d$, we have

$\frac{A}{1} = \frac{-2}{-2} = \frac{1}{1} \Rightarrow A = 1$

The distance between planes is $\frac{d}{\sqrt{1+4+1}} = \sqrt{6}$

$\therefore \quad |d| = 6$

**36. (6) :** Let $x$, $y$ and $z$ be the probabilities of $E_1$, $E_2$ and $E_3$. Then $x(1 - y)(1 - z) = \alpha$ ; $y(1 - x)(1 - z) = \beta$; $z(1 - x)(1 - y) = \gamma$

Also, $(1 - x)(1 - y)(1 - z) = p$

On dividing, we get, $\frac{x}{1-x} = \frac{\alpha}{p}$. $\quad \therefore \quad x = \frac{\alpha}{\alpha + p}$ etc.

$\frac{P(E_1)}{P(E_3)} = \frac{\frac{\alpha}{\alpha+p}}{\frac{\gamma}{\gamma+p}} = \frac{1 + \frac{p}{\gamma}}{1 + \frac{p}{\alpha}}$

Again, $(\alpha - 2\beta)p = \alpha\beta$ ; $(\beta - 3\gamma)p = 2\beta\gamma$

We have $\alpha p = \beta(\alpha + 2p)$ and $3\gamma p = \beta(p - 2\gamma)$

From above, $\frac{P(E_1)}{P(E_3)} = \frac{\frac{\alpha}{\alpha+p}}{\frac{\gamma}{\gamma+p}} = \frac{1 + \frac{p}{\gamma}}{1 + \frac{p}{\alpha}}$

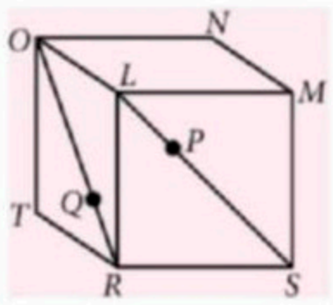
Also, $\frac{\alpha}{3\gamma} = \frac{\alpha + 2p}{p - 2\gamma} \Rightarrow \frac{\alpha + 2p}{\alpha} = \frac{p - 2\gamma}{3\gamma}$

$\Rightarrow \quad 1 + \frac{2p}{\alpha} = \frac{p}{3\gamma} - \frac{2}{3} \Rightarrow \frac{5}{3} = \frac{p}{3\gamma} - \frac{2p}{\alpha} \quad \Rightarrow \quad 5 = \frac{p}{\gamma} - \frac{6p}{\alpha}$

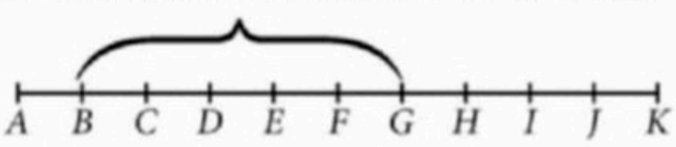
$\Rightarrow \quad 6\left(1 + \frac{p}{\alpha}\right) = 1 + \frac{p}{\gamma} \qquad \therefore \quad \frac{1 + \frac{p}{\gamma}}{1 + \frac{p}{\alpha}} = 6$

# OLYMPIAD CORNER

1. What is the minimum distance between the non-intersecting diagonals of adjacent faces of a cube of edge length 1? In other words, if, in the cube with $LM$ = 1 unit, points $P$ and $Q$ are chosen on the diagonals $LS$ and $OR$ so that the distance between them is as small as possible, then what is this distance?




2. A railway line is divided into 10 sections by the stations $A$, $B$, $C$, $D$, $E$, $F$, $G$, $H$, $I$, $J$ and $K$. The distance between $A$ and $K$ is 56 km. A trip along two successive sections never exceeds 12 km. A trip along three successive sections is at least 17 km. What is the distance between $B$ and $G$?




3. In a $\Delta ABC$, $\angle C = 2\angle B$. $P$ is a point in the interior of $\Delta ABC$ satisfying that $AP = AC$ and $PB = PC$. Show that $AP$ trisects $\angle A$.

4. Find the exact value of
$$\cos\left(\frac{2\pi}{17}\right)\cos\left(\frac{4\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\ldots\cos\left(\frac{16\pi}{17}\right).$$

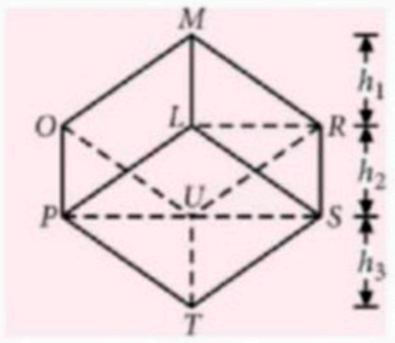
5. Let $p$ be a prime. Find all solutions in positive integers of the equation : $\frac{2}{a}+\frac{3}{b}=\frac{5}{p}$.

6. Let $x$ be a real number with $0 < x < \pi$. Prove that, for all natural numbers $n$, the sum
$$\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots+\frac{\sin(2n-1)x}{2n-1} \text{ is positive.}$$

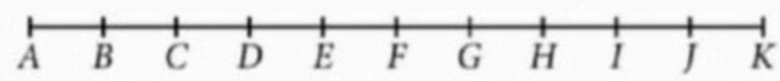
7. The feet of the angle bisectors of triangle $ABC$ are $X$, $Y$, and $Z$. The circumcircle of triangle $XYZ$ cuts off there segments from lines $AB$, $BC$, and $CA$. Prove that two of these segments' lengths add up to the third segment's length.

## SOLUTIONS

1. Stand the cube up on vertex $T$ so that the diagonal $MT$ is vertical ($U$ is the hidden vertex at the back). From the diagram, it is clear from symmetry that the point $OLR$ lie in a horizontal plane and the points $SUP$ lie in another. Since, the lines $LR$ and $PS$ lie in these planes, the distance between them is equal to the distance between the planes. Let $h_2$ be the vertical distance between the two planes. Let $h_1$ be the vertical distance between $M$ and the $OLR$ plane, which by symmetry is equal to the vertical distance between $T$ and the $PSU$ plane. But, since $LSTP$ is a square, its appearance in the diagram is a rhombus. Thus, $h_1 = h_2$ and so $h_1 = h_2 = \frac{1}{3}$ of the length $MT$. But $MT$ is a diagonal of the cube and is thus $\sqrt{3}$.



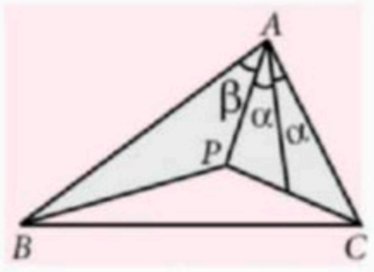

Thus the distance between the diagonals = the distance between the planes $\Rightarrow h_2 = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$

2. 


Now $\overline{AK} = 56$ and $\overline{AK} = \overline{AD} + \overline{DG} + \overline{GJ} + \overline{JK}$.
We know that $\overline{AD}, \overline{DG}, \overline{GJ} \geq 17$. Thus $\overline{JK} \leq 5$ to satisfy $\overline{AK} = 56$. We know $\overline{HK} \geq 17$ and since $\overline{JK} \leq 5$, $\overline{HJ} \geq 12$. But, we also know $\overline{HJ} \leq 12$. Thus, $\overline{HJ} = 12$.
Since, $\overline{HK} \geq 17$ and $\overline{HJ} = 12$, $\overline{JK} \geq 5$.
The only possibility is that $\overline{JK} = 5$.
Symmetrically, we find that $\overline{AB} = 5$ and $\overline{BD} = 12$.
Now, $\overline{DH} = \overline{AK} - \overline{AB} - \overline{BD} - \overline{HJ} - \overline{JK}$
$= 56 - 5 - 12 - 5 - 12 = 22$
Now, $\overline{GJ} \geq 17$ but $\overline{HJ} = 12$. Hence $\overline{GH} \geq 5$.
Since $\overline{DG} \geq 17$ and $\overline{DH} = \overline{DG} + \overline{GH} = 22$, we obtain $\overline{DG} = 17$ and $\overline{GH} = 5$.
Now, $\overline{BG} = \overline{BD} + \overline{DG} = 12 + 17 = 29$

**3.** Let $\angle PAC$ and $\angle BAP$ be $2\alpha$ and $\beta$ respectively. Then, since $\angle C = 2\angle B$, we deduce from $A + B + C = 180°$ that

$2\alpha + \beta + 3B = 180°$ ...(1)




The angles at the base of the isosceles triangle $PAC$ are each $90° - \alpha$. Also $\Delta BPC$ is isosceles, having base angles $C - (90° - \alpha) = 2B + \alpha - 90°$,

and so $\angle BPA = 180° - (\angle PBA + \angle BAP)$

$= 180° - [B - (2B + \alpha - 90°) + 180° - 2\alpha - 3B]$

$= 4B + 3\alpha - 90°$

As usual, let $a$, $b$ and $c$ denote the lengths of the sides $BC$, $AC$ and $AB$. By the Law of Cosines, applied to $\Delta BPA$, where $\overline{PA} = b$ and $\overline{PB} = \overline{PC} = 2b \sin\alpha$,

$c^2 = b^2 + (2b \sin\alpha)^2 - 2 \cdot b \cdot 2b \sin\alpha \cdot \cos(4B + 3\alpha - 90°)$,

so that $c^2 = b^2 [1 + 4\sin^2\alpha - 4 \sin\alpha \sin(4B + 3\alpha)]$ ...(2)

We now use the fact that $\angle C = 2\angle B$ is equivalent to the condition $c^2 = b(b + a)$.

Since $a = 2 \cdot \overline{PC} \cdot \cos(2B + \alpha - 90°)$

$= 4b \sin\alpha \sin(2B + \alpha)$, we have

$c^2 = b^2 [1 + 4 \sin\alpha \sin(2B + \alpha)]$ ...(3)

Therefore, from (2) and (3), we get

$b^2 [1 + 4\sin^2\alpha - 4 \sin\alpha \sin(4B + 3\alpha)]$

$= b^2 [1 + 4 \sin\alpha \sin(2B + \alpha)]$,

which simplifies to $\sin\alpha - \sin(4B + 3\alpha) = \sin(2B + \alpha)$.

Since, $\sin\alpha - \sin(4B + 3\alpha) = -2\cos(2B + 2\alpha) \sin(2B + \alpha)$, this equation may be rewritten as $\sin(2B + \alpha) \cdot [1 + 2\cos(2B + 2\alpha)] = 0$

Since, from (1), $2B + \alpha < 180°$, we must have $1 + 2\cos(2B + 2\alpha) = 0$, giving $\cos(2B + 2\alpha) = -1/2$; that is, $2B + 2\alpha = 120°$ ...(4)

Again from (1), $2B + 2\alpha < 180°$

Finally, we may eliminate $B$ between (1) and (4) to obtain $\alpha = \beta$. The result follows.

**4.** Let $I = \cos\frac{2\pi}{17}\cos\frac{4\pi}{17}\cos\frac{6\pi}{17}\ldots\cos\frac{16\pi}{17}$;

then, $I\sin\frac{2\pi}{17} = \frac{1}{2}\sin\frac{4\pi}{17}\cos\frac{4\pi}{17}\cos\frac{6\pi}{17}\ldots\cos\frac{16\pi}{17}$

$= \frac{1}{4}\sin\frac{8\pi}{17}\cos\frac{8\pi}{17}\cos\frac{6\pi}{17}\cos\frac{10\pi}{17}\cos\frac{12\pi}{17}\cos\frac{14\pi}{17}\cos\frac{16\pi}{17}$

$= \frac{1}{8}\sin\frac{16\pi}{17}\cos\frac{16\pi}{17}\cos\frac{6\pi}{17}\cos\frac{10\pi}{17}\cos\frac{12\pi}{17}\cos\frac{14\pi}{17}$

$= -\frac{1}{16}\sin\frac{2\pi}{17}\cos\frac{6\pi}{17}\cos\frac{10\pi}{17}\cos\frac{12\pi}{17}\cos\frac{14\pi}{17}$

Thus,

$I\sin\frac{6\pi}{17} = -\frac{1}{16}\sin\frac{6\pi}{17}\cos\frac{6\pi}{17}\cos\frac{10\pi}{17}\cos\frac{12\pi}{17}\cos\frac{14\pi}{17}$

$= -\frac{1}{32}\sin\frac{12\pi}{17}\cos\frac{12\pi}{17}\cos\frac{10\pi}{17}\cos\frac{14\pi}{17}$

$= \frac{1}{64}\sin\frac{10\pi}{17}\cos\frac{10\pi}{17}\cos\frac{14\pi}{17} = -\frac{1}{128}\sin\frac{14\pi}{17}\cos\frac{14\pi}{17}$

$= -\frac{1}{256}\sin\frac{28\pi}{17} = \frac{1}{256}\sin\frac{6\pi}{17}$. Therefore, $I = \frac{1}{256}$.

**5.** We have, $p(3a + 2b) = 5ab$; hence there are three cases to consider.

**Case I :** $p = 5$. Then we get

$3a + 2b = ab$ or $(a - 2)(b - 3) = 6 = 1.6 = 2.3$

Hence, all pairs of positive integers $(a, b)$ are $(3, 9)$, $(4, 6)$, $(5, 5)$ and $(8, 4)$.

**Case II :** $p$ divides $a$. Let $a = a_1 p$.

Hence, $3a_1 p = b(5a_1 - 2)$.

Then $p$ divides either $b$ or $5a_1 - 2$. If $b = b_1 p$, then $3a_1 = b_1(5a_1 - 2)$ or $(5a_1 - 2)(5b_1 - 3) = 6$,

which has only one solution $(a_1, b_1) = (1, 1)$, when $(a, b) = (p, p)$. Otherwise, $5a_1 - 2 = a_2 p$. Therefore,

$3\left(\frac{a_2 p + 2}{5}\right) = ba_2$ or $3a_2 p + 6 = 5ba_2$

Hence, $a_2$ divides 6.

If $a_2 = 1$, then $(a, b) = \left(\frac{p(p+2)}{5}, \frac{3(p+2)}{5}\right)$, but only if $p \equiv 3 \pmod 5$.

If $a_2 = 2$, then $(a, b) = \left(\frac{2p(p+1)}{5}, \frac{3(p+1)}{5}\right)$, but only if $p \equiv 4 \pmod 5$.

If $a_2 = 3$, then $(a, b) = \left(\frac{p(3p+2)}{5}, \frac{3p+2}{5}\right)$, but only if $p \equiv 1 \pmod 5$.

If $a_2 = 6$, then $(a, b) = \left(\frac{2p(3p+1)}{5}, \frac{3p+1)}{5}\right)$, but only if $p \equiv 3 \pmod 5$.

**Case III :** $p$ divides $b$. Let $b = b_1 p$.

Hence, $2b_1 p = a(5b_1 - 3)$

Then, $p$ divides either $a$ or $5b_1 - 3$. If $p$ divides $a$, then we have the same case ($p$ divides $a$ and $b$) as already considered above. Again $(p, p)$ is the (only) solution. Otherwise, $5b_1 - 3 = b_2 p$. Therefore,

$2\left(\frac{b_2 p + 3}{5}\right) = ab_2$ or $2b_2 p + 6 = 5ab_2$.

If $b_2 = 1$, then $(a, b) = \left(\frac{2(p+3)}{5}, \frac{p(p+3)}{5}\right)$, but only if $p \equiv 2 \pmod 5$.

If $b_2 = 2$, then $(a, b) = \left(\frac{2p+3}{5}, \frac{p(2p+3)}{5}\right)$, but only if $p \equiv 1 \pmod 5$.

If $b_2 = 3$, then $(a, b) = \left(\frac{2(p+1)}{5}, \frac{3p(p+1)}{5}\right)$, but only if $p \equiv 4 \pmod 5$.

If $b_2 = 6$, then $(a, b) = \left(\frac{2p+1}{5}, \frac{3p(2p+1)}{5}\right)$, but only if $p \equiv 2 \pmod 5$.

**Note :** If $p = 2$ or 3, this generates only two distinct solutions : (2, 2), (1, 6) for $p = 2$ and (3,3), (12, 2) for $p = 3$. If $p > 5$, then the three solutions are all distinct.

**6.** We use mathematical induction.

Let $S_n(x) = \sum_{k=1}^{n} \frac{\sin(2k-1)x}{(2k-1)}$.

$S_1(x) = \sin x > 0$ for $x \in (0, \pi)$. Thus the proposed inequality is true for $n = 1$. Let $S_r(x) > 0$ for $r = 1, 2, ..., n - 1$. We will deduce that $S_n(x) > 0$ for $x \in (0, \pi)$. Suppose that $S_n(x_0) \leq 0$ for some $x_0 \in (0, \pi)$, and that $S_n(x)$ attains its minimum at $x = x_0$.

Hence, $\frac{d}{dx}[S_n(x)]_{x=x_0} = 0$.

That is $S_n'(x_0) = \sum_{k=1}^{n} \cos((2k-1)x_0) = 0$,

so that $2 \sin x_0 S_n'(x_0) = \sum_{k=1}^{n} 2\cos((2k-1)x_0)\sin x_0$

$= \sum_{k=1}^{n} [\sin(2kx_0) - \sin((2k-2)x_0)] = \sin 2nx_0$

Thus $S_n'(x_0) = \frac{\sin 2nx_0}{2\sin x_0} = 0 \Rightarrow \sin 2nx_0 = 0$

Hence $x_0 \in \left\{\frac{\pi}{2n}, \frac{2\pi}{2n}, \frac{3\pi}{2n}, ..., \frac{(2n-1)}{2n}\right\}$.

It is easily verified that at each of these values $S_n(x_0) > 0$, a contradiction. Hence $S_n(x) > 0$ for $x \in (0, \pi)$.

**7.** We use signed distances throughout the problem, where $a = BC$, $b = CA$, and $c = AB$ are positive. Also, let the specified circle intersect $BC$ at $X$ and $P$, $CA$ at $Y$ and $Q$, and $AB$ at $Z$ and $R$ and let $x = PX$, $y = QY$, $z = RZ$. By the angle bisector theorem, $YA = \frac{bc}{(c+a)}$ and $AZ = \frac{bc}{(a+b)}$.

Hence, $QA = \frac{bc}{(c+a)+y}$ and $AR = \frac{bc}{(a+b)-z}$

Applying the "power of a Point theorem" to point $A$ and the specified circle gives

$$\frac{bc}{c+a}\left(\frac{bc}{c+a}+y\right) = \frac{bc}{a+b}\left(\frac{bc}{a+b}-z\right)$$

After multiplying through out by $\frac{a}{bc}$ and rearranging, we obtain

$$\frac{a}{c+a}y + \frac{a}{a+d}z = \frac{abc}{(a+b)^2} - \frac{abc}{(c+a)^2}$$

Similarly, we have

$$\frac{b}{a+b}z + \frac{b}{b+c}x = \frac{abc}{(b+c)^2} - \frac{abc}{(a+b)^2}$$

$$\frac{c}{b+c}x + \frac{c}{c+a}y = \frac{abc}{(c+a)^2} - \frac{abc}{(b+c)^2}$$

When we add these three equations, we get $x + y + z = 0$. It follows that two of the quantities $x, y, z$ are of the same sign and that the third is of the other sign. Hence, the sum of the absolute values of the former two quantities equals the absolute value of the latter, which is what we wished to prove.

## SAMURAI SUDOKU

ANSWER · JANUARY 2020

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| 6 | 4 | 7 | 8 | 5 | 1 | 9 | 3 | 2 | | | | 8 | 2 | 9 | 7 | 3 | 6 | 1 | 5 | 4 |
| 1 | 2 | 8 | 3 | 9 | 7 | 6 | 5 | 4 | | | | 6 | 5 | 3 | 4 | 1 | 8 | 9 | 2 | 7 |
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| 3 | 6 | 1 | 5 | 7 | 9 | 2 | 4 | 8 | 9 | 5 | 6 | 1 | 3 | 7 | 2 | 8 | 5 | 4 | 9 | 6 |
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| 6 | 3 | 7 | 1 | 9 | 4 | 8 | 5 | 2 | 7 | 3 | 4 | 6 | 1 | 9 | 3 | 8 | 5 | 2 | 7 | 4 |
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| 7 | 8 | 2 | 6 | 4 | 5 | 3 | 9 | 1 | | | | 5 | 3 | 4 | 6 | 7 | 8 | 9 | 1 | 2 |
| 9 | 6 | 4 | 3 | 1 | 8 | 5 | 2 | 7 | | | | 1 | 6 | 8 | 2 | 9 | 3 | 4 | 5 | 7 |
| 5 | 1 | 6 | 7 | 2 | 9 | 4 | 8 | 3 | | | | 4 | 7 | 1 | 8 | 5 | 9 | 6 | 2 | 3 |
| 8 | 7 | 3 | 4 | 5 | 1 | 2 | 6 | 9 | | | | 3 | 8 | 6 | 1 | 4 | 2 | 7 | 9 | 5 |
| 4 | 2 | 9 | 8 | 6 | 3 | 1 | 7 | 5 | | | | 9 | 2 | 5 | 7 | 3 | 6 | 8 | 4 | 1 |

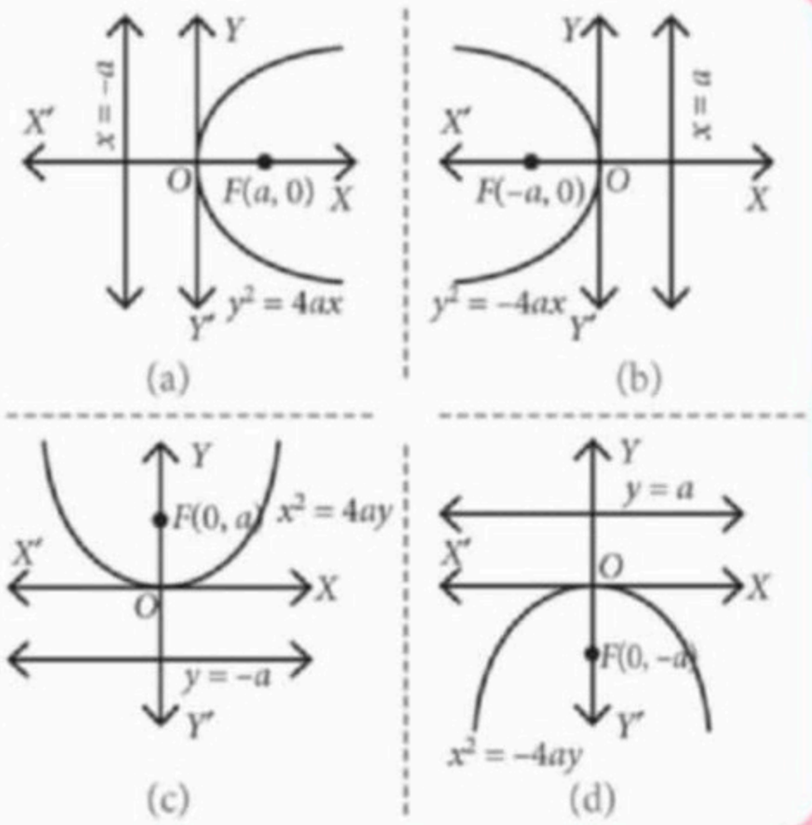
CONCEPT MAP

# PARABOLA

Class XI

## Definition

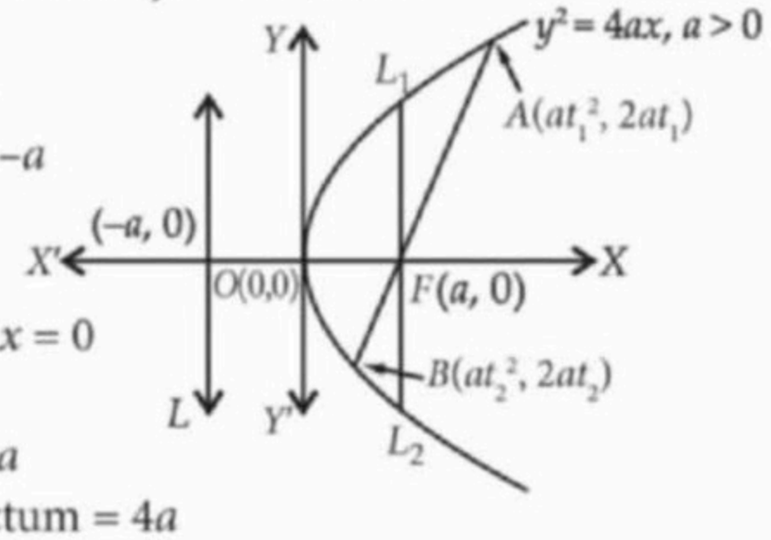
ola is the of point moves in a such that its ce from a point is equal its distance a fixed line. fixed point called the . The fixed is called the rix.




## Terms Related to Parabola in Standard Form

Standard form of parabola is $y^2 = 4ax$, $a > 0$

- Focus ($F$) : $(a, 0)$
- Vertex (O) : $(0, 0)$
- Directrix ($L$) : $x = -a$
- Axis : $y = 0$
- Equation of tangent at vertex : $x = 0$
- Equation of latus rectum, $L_1L_2$ : $x = a$
- Length of latus rectum = $4a$
- Eccentricity, $e = 1$
- Parametric equations: $x = at^2$, $y = 2at$




## Equation of Tangent

| Curve | Point | Equation of Tangent |
|---|---|---|
| $= 4ax$ | $(x_1, y_1)$ | $yy_1 = 2a(x + x_1)$ (Point form) |
| | $(at^2, 2at)$ | $ty = x + at^2$ (Parametric form) |
| | $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ | $y = mx + \frac{a}{m}$ (Slope form) |

## Conditions of Tangency of a Line to the Parabola

| Parabola | Lines | Conditions |
|---|---|---|
| $y^2 = 4ax$ | $y = mx + c$ | $c = \frac{a}{m}$ |
| | $x\cos\alpha + y\sin\alpha = p$ | $p = -a\tan\alpha\sin\alpha$ |
| | $lx + my + n = 0$ | $ln = am^2$ |

## Equation of Normal

| Curve | Point | Equation of Normal |
|---|---|---|
| $= 4ax$ | $(x_1, y_1)$ | $y - y_1 = -\frac{y_1}{2a}(x - x_1)$ (Point form) |
| | $(at^2, 2at)$ | $y + tx = at^3 + 2at$ (Parametric form) |
| | $(am^2, -2am)$ | $y = mx - 2am - am^3$ (Slope form) |

## Some Important Results

- Locus of the point of intersection of perpendicular tangents to the parabola $y^2 = 4ax$ is its directrix, $x = -a$.
- Equation of chord of parabola $y^2 = 4ax$ bisected at point $(x_1, y_1)$ is given by $yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$.
- The locus of the middle points of the focal chord of parabola $y^2 = 4ax$ is $y^2 = 2a(x-a)$.
- If the normals at $t_1$ and $t_2$ meet the parabola $y^2 = 4ax$, then $t_2 = -t_1 - \frac{2}{t_1}$.
- The semi-latus rectum of a parabola is the H.M. between the segments of any focal chord of a parabola.
- Normals at the end points of latus rectum of the parabola intersect at right angle on the axis.

# VECTOR ALGEBRA

CONCEP MAP

**Class XII**

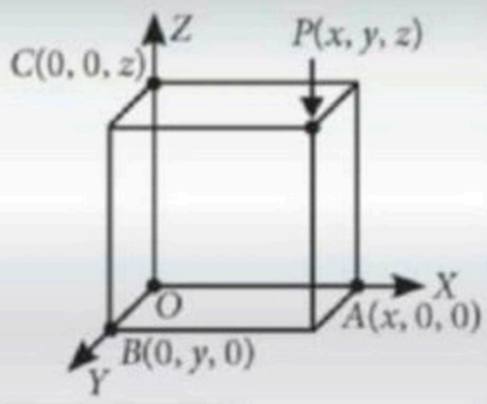
## Vectors

The physical quantities that has magnitude as well as direction are called vectors. For example: velocity, acceleration, weight, force, etc.

### Components of a Vector in 3D

Let $P(x, y, z)$ be a point.

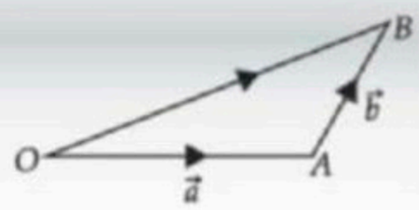
$x = OA$, $y = OB$, $z = OC$ and $\hat{i}, \hat{j}, \hat{k}$ be unit vectors, then $\overrightarrow{OA} = x\hat{i}$, $\overrightarrow{OB} = y\hat{j}$, $\overrightarrow{OC} = z\hat{k}$ are components of $\overrightarrow{OP}$ along $x$, $y$, $z$ axes respectively.



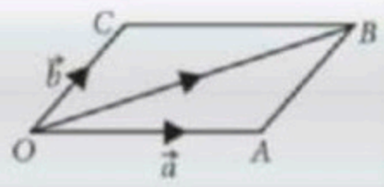

$\therefore \overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ and $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

## Addition of Vectors

- **Triangle Law :** In $\Delta OAB$, let position vectors of $\overrightarrow{OA}$ and $\overrightarrow{AB}$ be $\vec{a}$ and $\vec{b}$ respectively. Then, $\overrightarrow{OB} = \vec{a} + \vec{b}$




- **Parallelogram Law :** In parallelogram $OABC$, let position vectors of $\overrightarrow{OA}$ and $\overrightarrow{OC}$ be $\vec{a}$ and $\vec{b}$ respectively. Then, $\overrightarrow{OB} = \vec{a} + \vec{b}$




### Properties of Scalar Triple Product

- $[\vec{a}\times\vec{b}]\cdot\vec{c} = \vec{a}\cdot[\vec{b}\times\vec{c}]$
- $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$
- $[\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}] = -[\vec{b}\,\vec{a}\,\vec{c}] = -[\vec{c}\,\vec{b}\,\vec{a}]$
- $[\alpha\vec{a}\ \beta\vec{b}\ \gamma\vec{c}] = \alpha\beta\gamma[\vec{a}\,\vec{b}\,\vec{c}]$
- $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$
- $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0$
- $[\vec{a}\,\vec{b}\,\vec{c}] = 0$, if any two of the vectors are parallel.
- Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar iff $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

  $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

- $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$

## Multiplication of Vectors

### Product of Two Vectors

Let $\vec{a}, \vec{b}$ be two non-zero vectors, inclined at an angle $\alpha$

- Scalar or dot product, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\alpha$
- Vector or cross product, $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\alpha\hat{n}$, $\hat{n}$ is the unit vector perpendicular to the plane and $\vec{b}$.

### Properties of Scalar Product

- $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
- $\vec{a}\cdot(\vec{b}\pm\vec{c}) = \vec{a}\cdot\vec{b} \pm \vec{a}\cdot\vec{c}$
- $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$
- $\vec{a}\cdot\vec{a} = |\vec{a}|^2$
- $\vec{a}\cdot(\alpha\vec{b}) = \alpha(\vec{a}\cdot\vec{b}$
- $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} =$
- $\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = \hat{j}\cdot\hat{i} = \hat{k}\cdot\hat{i} = \hat{k}\cdot\hat{j} = 0$
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

  $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

### Properties of Vector Product

- $\vec{b}\times\vec{a} = -\vec{a}\times\vec{b}$
- $\vec{a}\times\vec{a} = 0$
- $\vec{a}\times(\vec{b}\pm\vec{c}) = \vec{a}\times\vec{b} \pm \vec{a}\times\vec{c}$
- $(\alpha\vec{a})\times(\beta\vec{b}) = (\alpha\beta)($
- $\vec{a}\times\vec{b} = 0 \Leftrightarrow \vec{a} \parallel \vec{b}$
- $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$
- $\hat{i}\times\hat{j} = -\hat{j}\times\hat{i} = \hat{k}$, $\hat{j}\times\hat{k} = -\hat{k}\times\hat{j} = \hat{i}$, $\hat{k}\times\hat{i} = -\hat{i}\times\hat{k} =$
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- If $\vec{a}$ and $\vec{b}$ are adjacent sides of a triangle, then a triangle $= \frac{1}{2}|\vec{a}\times\vec{b}|$.

### Scalar Triple Product

Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors, then the product $\vec{a}\cdot(\vec{b}\times\vec{c})$ is c scalar triple product of vectors $\vec{a}, \vec{b}, \vec{c}$. It represents the vo of the parallelepiped, whose coterminous edges are $\vec{a}, \vec{b}, \vec{c}$ it is denoted by $[\vec{a}\,\vec{b}\,\vec{c}]$.

# Challenging PROBLEMS ON JEE Advanced

Exam on 17th May, 2020

## Single Option Correct Type

**1.** If a curve is represented parametrically by the equations $x = 4t^3 + 3$, $y = 4 + 3t^4$ and $\dfrac{\left(\dfrac{d^2x}{dy^2}\right)}{\left(\dfrac{dx}{dy}\right)^n}$ is a constant, then the value of $n$ is

(a) 3 (b) 4 (c) 5 (d) 6

**2.** A circle with centre $N$ having double contact with parabola $y^2 = (2+\sqrt{3})x$ passes through its focus $S$. If tangent at their point of contact $P$ intersect its axis at $R$, then radius of inscribed circle of triangle $PSR$ is

(a) $2\sqrt{3}$ (b) $\dfrac{\sqrt{3}}{4}$ (c) $\sqrt{3}$ (d) $\dfrac{\sqrt{3}}{2}$

**3.** From a point $P$ perpendicular tangents $PQ$ and $PR$ are drawn to ellipse $x^2 + 4y^2 = 4$, then locus of circumcenter of triangle $PQR$ is

(a) $x^2 + y^2 = \dfrac{16}{5}(x^2 + 4y^2)^2$

(b) $x^2 + y^2 = \dfrac{5}{16}(x^2 + 4y^2)^2$

(c) $x^2 + 4y^2 = \dfrac{16}{5}(x^2 + y^2)^2$

(d) $x^2 + 4y^2 = \dfrac{5}{16}(x^2 + y^2)^2$

**4.** $M$ is the mid-point of side $AB$ of equilateral triangle $ABC$. $P$ is a point on $BC$ such that $AP + PM$ is minimum. If $AB = 20$, then $AP + PM$ is

(a) $10\sqrt{7}$ (b) $10\sqrt{3}$ (c) $10\sqrt{5}$ (d) 10

**5.** A sequence of numbers $A_n, n = 1, 2, 3, \ldots$ is defined as follows $A_1 = \dfrac{1}{2}$ and for each $n \geq 2$, $A_n = \left(\dfrac{2n-3}{2n}\right)A_{n-1}$, then

(a) $\sum_{k=1}^{n} A_k = 1, \forall n \geq 1$ (b) $\sum_{k=1}^{n} A_k > 1, \forall n > 1$

(c) $\sum_{k=1}^{n} A_k < 1, \forall n \geq 1$ (d) $\sum_{k=1}^{n} A_k > \dfrac{1}{2}, \forall n \geq 1$

**6.** For the natural numbers $n$ and $k$, let

$$T(n,k) = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \ldots + \frac{1}{(n+k)!},$$

then which of the following is false?

(a) $T(n,k) \leq \dfrac{1}{n}\left(\dfrac{1}{(n)!} - \dfrac{1}{(n+k)!}\right)$

(b) $\lim_{k\to\infty} T(n,k) \leq \dfrac{1}{n!}$ (c) $T(n,n) < \dfrac{1}{(n-1)!}$

(d) $T(2, 2)$ is equal to 2/3

**7.** If $\alpha$ is a root of the equation $x^{11} - x^{10} + x^8 - x^7 + x^5 - x^4 + x^2 - x - 20 = 0$, then $\alpha^{12}$ is

(a) $= 61$ (b) $> 61$ (c) $< 61$ (d) can't say

## Multiple Options Correct Type

**8.** If the function

$f(x) = \left(\sqrt{4-x^2} - 3\right)^2 + \left(\sqrt{4-x^2} + 1\right)^3$, then which of the following is/are correct? ([·] denotes G.I.F.)

(a) $f(x)$ has greatest value 28

(b) $f(x)$ has least value $k$, where $[k] = 9$

(c) The slope of tangent of $f(x)$ at $x = 1$ equals $-(8+2\sqrt{3})$

(d) $f'(2) = 0$

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**9.** If $|f(x)| = |ax^2 + bx + c| \le 1$ for $|x| \le 1$ $(a, b, c \in R)$ and $\frac{8}{3}a^2 + 2b^2$ is maximum, then

(a) $b = 0$ (b) $|a + c| = 1$

(c) $ac = 1$ (d) $\left|\frac{c}{a}\right| = \frac{1}{2}$

**10.** Let $f(x) = \begin{cases} 0, x = 0 \\ (1-x)^{\alpha} x^{\beta}(1 - \cos 2\pi x), 0 < x < 1. \\ 0, x = 1 \end{cases}$

If Rolle's theorem is applicable to $f(x)$ for $x \in [0, 1]$, then

(a) $\alpha \in R$ (b) $\beta \in R$

(c) $\beta \in (-2, \infty)$ (d) $\alpha \in (-2, \infty)$

**11.** Let $f : R \to R$ be a differentiable function which satisfies $f(x) = \int_0^x |x - f(x)|\, dx \ \forall\ x \in R$, then which of the following is/are correct?

(a) $f(x)$ is a monotonic function

(b) $f(x)$ is an onto function

(c) $f(x)$ is an odd function

(d) $y = x$ is tangent to $f(x)$

**12.** Consider the given expression,

$$\cos^6\beta + \cos^6\alpha - \cos^6\beta\cos^2\alpha$$

$$L = \frac{-\cos^6\alpha\cos^2\beta}{\cos^4\alpha\cos^4\beta - \cos^6\alpha\cos^4\beta - \cos^4\alpha\cos^6\beta + \cos^6\alpha\cos^6\beta}$$

where, $\alpha, \beta \ne \frac{n\pi}{2}, n \in I$. The minimum value of $\left[\frac{L}{2} + 1\right]$ is $\lambda/\mu$ (where $\lambda$ & $\mu$ are relatively prime numbers and $[\cdot]$ is greatest integer function), then

(a) $\lambda + \mu = 5$ (b) $\lambda + \mu = 6$

(c) $\lambda - \mu = 4$ (d) $\lambda - \mu = 3$

## Comprehension Type

### Paragraph for Q. No. 13 and 14

A parabola touches the bisectors of the angle obtained by the lines $x + 2y + 3 = 0$ and $2x + y + 3 = 0$ at the points (1, 1) and (0, –2).

**13.** The equation of the directrix is

(a) $3x + y + 4 = 0$ (b) $x - 3y - 2 = 0$

(c) $8x - y + 7 = 0$ (d) $3x + y - 2 = 0$

**14.** The length of latus rectum is

(a) $\frac{16}{5^{3/2}}$ (b) $\frac{16\sqrt{2}}{5}$ (c) $\frac{16\sqrt{5}}{2^{3/2}}$ (d) $\frac{16\sqrt{2}}{5^{3/2}}$

## Numerical Value Type

**15.** $\lim_{x \to 0} \left\{ \sum_{r=0}^{n} (-1)^r {}^nC_r \left( \sum_{k=0}^{n-r} {}^{(n-r)}C_k\, x^k 2^k \right) (x^2 - x)^r \right\}^{1/x} = e^{\lambda n}$,

then $\lambda =$

**16.** The least value of '$a$' such that $f(x) \ge 0\ \forall\ x \in R$ where $f(x) = x^4 - 2x^3 + ax^2 - 2x + 1$, is equal to

**17.** Let $A = \left(\frac{1}{2}, 0\right), B = \left(\frac{3}{2}, 0\right), C = \left(\frac{5}{2}, 0\right)$ be three points on $xy$-plane. $P$ is another point on the same plane satisfying $\max\{PA + PB, PB + PC\} < 2$. If the area of the region of $P$ is $\sqrt{3}\left(\frac{\pi}{a} - \frac{\sqrt{3}}{b}\right)$, then $a + b =$

**18.** Let $OHFM$ is a rectangle with $OM = 5$, $FM = 11$, $ABC$ is a triangle with orthocentre $H$, circumcentre $O$. $M$ is the mid-point of side $BC$ of $\Delta ABC$ and $F$ is the foot of the altitude from vertex $A$ to the side $BC$, then the area of the triangle $ABC$ is $k$, where the sum of the digits in $k$ is

## SOLUTIONS

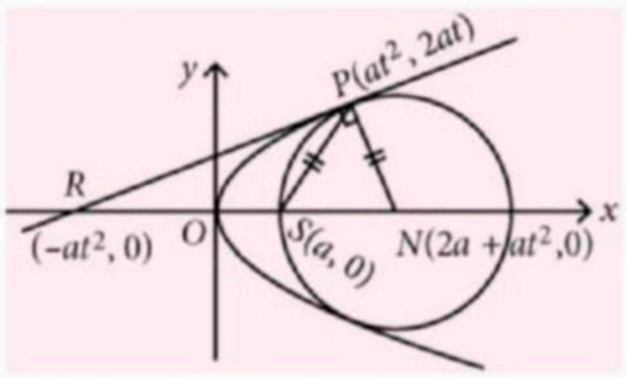
**1. (c) :** $\frac{dx}{dy} = \frac{dx/dt}{dy/dt} = \frac{12t^2}{12t^3} = \frac{1}{t}$

Now, $\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dt}\left(\frac{1}{t}\right)\frac{dt}{dy} = \left(\frac{-1}{t^2}\right)\frac{1}{12t^3} = \frac{-1}{12t^5}$

Now, $\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^n} = \frac{\left(\frac{-1}{12t^5}\right)}{\left(\frac{1}{t^n}\right)} = \frac{-t^{n-5}}{12} = \text{constant}$

Hence, $n = 5$

**2. (d) :**




$N$ is centre of circle $\therefore$ $SN = SP = PN$ (radius)

$\Rightarrow$ $\angle PNS = 60°$

Now, in $\Delta PSR$, $\angle R = 30°$, $PS = SR$

$\Rightarrow$ $\tan 30° = \frac{1}{t} \Rightarrow t = \sqrt{3}$

$\therefore$ $P(3a, 2\sqrt{3}a), S(a, 0), R(-3a, 0)$

Now, $r=\dfrac{\Delta}{s}=\dfrac{\frac{1}{2}\cdot RS\cdot PS\cdot\frac{\sqrt{3}}{2}}{\frac{1}{2}(RS+PS+RP)}=\dfrac{\sqrt{3}}{2}$

**3. (b) :** $P$ Lies on $x^2+y^2=5$. Let $P(\sqrt{5}\cos\theta,\sqrt{5}\sin\theta)$. Since $\Delta QPR$ is right angled at point $P$, circumcenter is mid-point of $QR$.

$QR$ is chord of contact w.r.t point $P$.

$\therefore$ Its equation is $\dfrac{x\sqrt{5}\cos\theta}{4}+\dfrac{y\sqrt{5}\sin\theta}{1}=1$ ...(i)

Also, equation of the chord $QR$, which is bisected at point $(h, k)$ is

$$\frac{xh}{h^2+4k^2}+\frac{4ky}{h^2+4k^2}=1 \quad ...(ii)$$

Comparing coefficients of equations (i) and (ii), we get

$$\frac{\sqrt{5}\cos\theta}{4}=\frac{h}{h^2+4k^2} \text{ and } \sqrt{5}\sin\theta=\frac{4k}{h^2+4k^2}$$

$\therefore$ Required locus is $x^2+y^2=\dfrac{5}{16}(x^2+4y^2)^2$

**4. (a) :** Take the reflection of $\Delta ABC$ in $BC$.

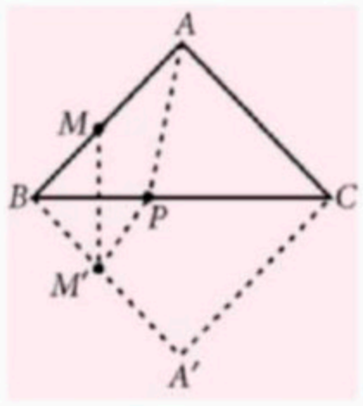
So, $PM = PM'$

$PA + PM = PA + PM'$

Now, it is minimum when $M'$ $PA$ lies in a line.

On applying cosine rule in triangle $ABM'$, we will get

$AM'=10\sqrt{7}$




**5. (c) :** $A_n=\left(\dfrac{2n-3}{2n}\right)A_{n-1}$

$(2n-1)A_n-(2n-3)A_{n-1}=-A_n$

$(2n-3)A_{n-1}-(2n-5)A_{n-2}=-A_{n-1}$

.....

.....

$3A_2-A_1=-A_2$

Adding all above equations, we get

$(2n-1)A_n-A_1=-(A_2+A_3+.....+A_n)$

$\Rightarrow A_1+A_2+.....+A_n=2A_1-(2n-1)A_n$

$=1-(2n-1)A_n$

$<1$

$\therefore (2n-1)A_n=A_1-(A_2+A_3+.....+A_n)<1/2$

**6. (d) :** For $i\le j\le k$

$$\frac{n}{(n+j)!}\le\frac{n+j-1}{(n+j)!}=\frac{1}{(n+j-1)!}-\frac{1}{(n+j)!}$$

$$\Rightarrow n\sum_{j=1}^{k}\frac{1}{(n+j)!}\le\frac{1}{n!}-\frac{1}{(n+k)!}$$

$$\lim_{k\to\infty}T(n,k)\le\frac{1}{n.n!}\le\frac{1}{n!}$$

$$T(n,n)=\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\frac{1}{(n+3)!}+....+\frac{1}{(2n)!}<\frac{1}{n!}+\frac{1}{n!}+\frac{1}{n!}+...+\frac{1}{n!}=\frac{1}{(n-1)!}$$

$T(2,2)<1$

**7. (b) :** $(x^{11}+x^8+x^5+x^2)-(x^{10}+x^7+x^4+x)=20$

$$\Rightarrow \frac{x^2(x^{12}-1)}{x^3-1}-x\frac{(x^{12}-1)}{x^3-1}=20$$

$$\Rightarrow x\left(x^{12}-1\right)=20\left(x^2+x+1\right)$$

$$\Rightarrow x^{12}=20\left(x+\frac{1}{x}+1\right)+1$$

**8. (a,b,c):** $x\in[-2,2]\Rightarrow a\in[0,2]$, where $a=\sqrt{4-x^2}$

$f(a)=(a-3)^2+(a+1)^3$

$f'(a)=3a^2+8a-3=(3a-1)(a+3)$

$\therefore f_{min}=f(1/3)=\dfrac{64}{9}+\dfrac{64}{27}=\dfrac{256}{27}$

$f_{max}=f(-3)=28$

$\dfrac{dy}{dx}=(6+8\sqrt{3})\dfrac{1}{2\sqrt{4-1}}(-2\times1)$ $(\because x=1\Rightarrow a=\sqrt{3})$

$=-\left(\dfrac{6+8\sqrt{3}}{\sqrt{3}}\right)=-(8+2\sqrt{3})$

**9. (a, b, d) :** We have, $\dfrac{8}{3}a^2+2b^2=\dfrac{2}{3}[4a^2+3b^2]$

$=\dfrac{2}{3}[4(a^2+b^2)-b^2]$

$|f(1)|\le1,\ |f(0)|\le1,\ |f(-1)|\le1$

$|f(1)-f(0)|\le2\Rightarrow|a+b|\le2$

$|f(-1)-f(0)|\le2\Rightarrow|a-b|\le2$

$4(a^2+b^2)-b^2=2((a+b)^2+(a-b)^2)-b^2\le2\times8-b^2$

$|a+b|=2,\ |a-b|=2$

When $f(1)=1, f(0)=-1, f(-1)=1$

or $f(1)=-1,\ f(0)=1, f(-1)=-1$

In the second case, we get $b=0, a=-2, c=1$

**10. (c, d) :** Since, $f(0)=0, f(1)=0$

$f(x)$ is continuous at $x=0\Rightarrow f(0^+)=f(0)$

$$\Rightarrow 0=\lim_{x\to0^+}(1-x)^\alpha x^\beta(2\sin^2\pi x)$$

$$=\lim_{x\to0^+}2\pi^2(1-x)^\alpha x^2\cdot x^\beta=\lim_{x\to0^+}2\pi^2(1-x)^\alpha x^{\beta+2}$$

$$\Rightarrow 2\pi^2\lim_{x\to0^+}x^{\beta+2}=0$$

$\Rightarrow \beta + 2 > 0 \Rightarrow \beta > -2$

But for continuity at $x = 1$

$$f(1) = f(1^-) \Rightarrow 0 = \lim_{h\to 0^+} h^\alpha (1-h)^\beta (1-\cos(2\pi - 2\pi h))$$

$$\Rightarrow 0 = \lim_{h\to 0^+} h^\alpha (1-h)^\beta (2\sin^2 h\pi)$$

$$\Rightarrow \lim_{h\to 0^+} h^\alpha (1-h)^\beta \frac{2(\sin^2 \pi h)}{(\pi h)^2}(\pi h)^2 = 0$$

$$\Rightarrow 2\pi^2 \lim_{h\to 0^+} h^{\alpha+2} = 0 \Rightarrow \alpha + 2 > 0 \Rightarrow \alpha > -2$$

Also, $f(x)$ is derivable in (0, 1).

Therefore Rolle's theorem is applicable if $\alpha > -2, \beta > -2$.

**11. (a, b, c) :** $f'(x) = |x - f(x)|$

$\Rightarrow f'(x) + f(x) = x$ if $x \geq f(x)$

and $f'(x) - f(x) = -x$ if $x \leq f(x)$

$\Rightarrow f(x) = (x - 1) + e^{-x}$ if $x \geq f(x)$

and $f(x) = (x + 1) - e^x$ if $x \leq f(x)$

$$\Rightarrow f(x) = \begin{cases} x-1+e^{-x}, & x \geq 0 \\ x+1-e^x, & x \leq 0 \end{cases}$$

**12. (b, c) :**

$$L = \frac{\cos^6\beta + \cos^6\alpha - \cos^6\beta\cos^2\alpha - \cos^6\alpha\cos^2\beta}{\cos^4\alpha\cos^4\beta - \cos^6\alpha\cos^4\beta - \cos^4\alpha\cos^6\beta + \cos^6\alpha\cos^6\beta}$$

$$= \frac{\sec^4\alpha}{\tan^2\beta} + \frac{\sec^4\beta}{\tan^2\alpha}$$

Let $a = \tan^2\alpha$ and $b = \tan^2\beta$

Then, $L = \frac{(a+1)^2}{b} + \frac{(b+1)^2}{a}$ $(a, b \geq 0)$

$$\geq 4\left[\frac{a^2}{b}\times\frac{1}{b}\times\frac{b^2}{a}\times\frac{1}{a}\right]^{\frac{1}{4}} + 4\left[\frac{a}{b}\times\frac{b}{a}\right]^{\frac{1}{2}} = 8$$

**(13–14) : 13. (a)** **14. (d)**

The equation of bisectors are $x - y = 0$, $x + y + 2 = 0$, whose point of intersection is $P(-1, -1)$.

Focus $S$, is the foot of perpendicular from $P$ to $MN$

$M(1, 1)$, $N(0, -2)$ *i.e.*, $S \equiv \left(\frac{1}{5}, \frac{-7}{5}\right)$

$SM = 4\sqrt{\frac{2}{5}}$, $NS = \sqrt{\frac{2}{5}}$, whose harmonic mean is semi-latus rectum.

$\therefore$ Latus rectum $= 2\left(\frac{2SM\cdot NS}{SM+NS}\right) = \frac{16\sqrt{2}}{5^{3/2}}$

Since, $\angle MPN = \frac{\pi}{2}$, hence circle on $MN$ diameter touches the directrix at $P$.

$\therefore$ The tangent at $P$ to circle $(x - 0)(x - 1) + (y - 1)(y + 2) = 0$ is the directrix, which is $3x + y + 4 = 0$.

**15. (3) :** Let $I = \sum_{r=0}^{n} (-1)^r\ {}^nC_r\ (1+2x)^{n-r}\ (x^2 - x)^r \Big\}^{1/x}$

$$= (1 + 2x - x^2 + x)^{n/x}$$

$$\therefore\ e^{\lambda n} = e^{\lim_{x\to 0}(3x-x^2)\frac{n}{x}} = e^{3n}$$

**16. (2) :** $f(x) = (x^2 - x)^2 + (x - 1)^2 + (a - 2)x^2$

$= (x - 1)^2(x^2 + 1) + (a - 2)x^2 \geq 0$ if $a \geq 2$

**17. (7) :** Required area is bounded between two ellipse

$\frac{(x-1)^2}{1} + \frac{y^2}{3/4} = 1$ and $\frac{(x-2)^2}{1} + \frac{y^2}{3/4} = 1$.

$\therefore$ Required area

$$= 4\int_{3/2}^{2} \frac{\sqrt{3}}{2}\sqrt{1-(x-1)^2}\,dx = \sqrt{3}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

$\therefore$ $a = 3$, $b = 4 \Rightarrow a + b = 7$

**18. (3) :** Take $O$ as the origin, $OM$ is negative $y$-axis.

Then $M = (0, -5)$, $F = (-11, -5)$, $A = (-11, 10)$

Let $B = (-x, -5)$ and $C = (x, -5)$

Now, $OB = OA \Rightarrow x^2 + 25 = 121 + 100$

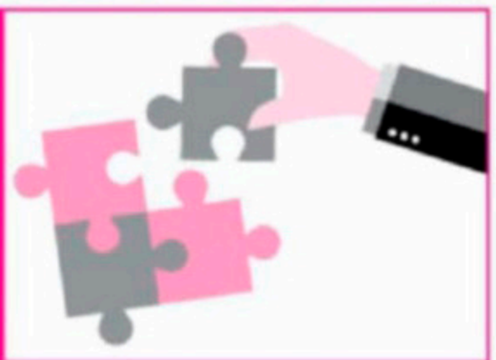
$\Rightarrow x^2 = 196 \Rightarrow x = 14$ $\therefore$ $BC = 28$

So, area of $\Delta ABC = \frac{1}{2}\times 28\times 15 = 210 = k$

$\therefore$ Sum of digits of $k = 3$

# PUZZLE CORNER

ANSWER · JANUARY 2020



| | | | | | |
|---|---|---|---|---|---|
| 6× 2 | 15× 5 | 3 | 10+ 4 | 6 | 6+ 1 |
| 3 | 24× 4 | 11+ 2 | 4− 5 | 1 | 6 |
| 1− 6 | 1 | 5 | 5+ 3 | 2 | 2− 4 |
| 5 | 6 | 4 | 12× 1 | 900× 3 | 2 |
| 8× 4 | 2 | 18× 1 | 6 | 5 | 3 |
| 1 | 3 | 6 | 2 | 4 | 5 |

# The Magic of Vedic Mathematics

## MULTIPLICATION WITH A SERIES OF 9's

In the whole number series, 9 is the most interesting number therefore the multiplication with either of 99, 999, 9999, ... so on is also very interesting. For this we use the sub sutra

"EkaDhikena Pūrvena"

which means

**"By one less than the one before"**

along with the sutra

"Nikhilam Navatah caramam Dasatah"

which means

**"All from nine and last from ten"**

On basis of number of digits in Multiplicand and Multiplier, the following three cases arise:

**Case-I :** When the number of digits in the multiplicand (number to be multiplied) and the multiplier are the same and all the digits of the multiplier are 9's.

**Steps to follow :** Divide the answer in two parts, the Left Hand Side (L.H.S.) and the Right Hand Side (R.H.S.)

**Step-1:** L.H.S. of the answer is one less than the multiplicand (number to be multiplied).

**Step-2 :** R.H.S. is the complement of the multiplicand number or subtract each digit of the L.H.S. of the answer individually from 9.

**Note :** Complement of a given number is the difference of the number from 10,100 or 1000 etc., having no number of digits in it.

**For example,** complement of 4 = 10 – 4 = 6,

complement of 68 = 100 – 68 = 32.

**Example 1 :** 342 × 999

**Step-1 :** L.H.S. is one less than the multiplicand.

Here, 342 – 1 = 341

$$\begin{array}{r|l} 342 \times 999 & \\ \hline 341 & \end{array}$$

**Step-2 :** R.H.S. is the complement of the multiplicand using "All from 9 and last from 10" rules.

Here, complement of 342 is 658

Or subtract each digit of the L.H.S of the answer 341 individually from 9 to get 658.

$$\begin{array}{r|l} 342 \times 999 & \\ \hline 341 & 658 \end{array}$$

∴ Answer is 341658.

**Example 2 :** 536874 × 999999

**Step-1 :** L.H.S. is 536874 – 1 = 536873

$$\begin{array}{r|l} 536874 \times 999999 & \\ \hline 536873 & \end{array}$$

**Step-2 :** R.H.S. is the complement of 536874 = 463126

$$\begin{array}{r|l} 536874 \times 999999 & \\ \hline 536873 & 463126 \end{array}$$

∴ Answer is 536873463126.

**Case-II :** When the number of digits in the multiplier are more than the digit in the multiplicand and all the digits of the multiplier are 9's.

**Steps to follow:**

**Step-1 :** Make the number of digits of the multiplicand equal to the number of digits of the multiplier by adding zeroes before the number.

**Step-2 :** L.H.S. of the answer is one less than the multiplicand.

**Step-3 :** R.H.S. of the answer is the complement of the number (the one formed by adding the zeroes).

Or subtract each digit of the L.H.S. of the answer individually from 9.

**Example 1 :** 23 × 999

**Step-1 :** Since the multiplicand had 2 digits and multiplier has 3 digits, make the multiplicand as 023, so that the digits of the multiplicand are equal to the digits of the multiplier.

23 × 999 = 023 × 999

By : Vipin Kaur, Govt Mathematics Lecturer, Hoshiarpur

**Step-2 :** L.H.S. is one less than the multiplicand
Here, 823 − 1 = 822

| 823 × 999 |
|---|
| 822 \| |

**Step-3 :** R.H.S. is the complement of 823 = 977
Or subtract each digit of the L.H.S. of the answer 1822 individually from 9 to get 977.

| 823 × 999 |
|---|
| 822 \| 977 |

∴ Answer is 329977.

**Example 2 :** 548 × 99999
**Step-1 :** Since the multiplicand has 3 digits and multiplier has 5 digits, make the multiplicand as 00548

548 × 99999 = 00548 × 99999

**Step-2 :** L.H.S. is 00548 − 1 = 00547

| 00548 × 99999 |
|---|
| 00547 \| |

**Step-3:** R.H.S. is the complement of 00548 = 99452

| 00548 × 99999 |
|---|
| 00547 \| 99452 |

∴ Answer is 54799452.

**Case III :** When the number of digits in the multiplier are less than the number of digits in the multiplicand and all the digits in the multiplier are 9's.
**Steps to follow:**
**Step-1 :** Divide the multiplicand in two parts such that the right part has the same number of digits in the multiplier.
**Step-2 :** Increase the left part by one and subtract it from the multiplicand to form the L.H.S of the answer. The R.H.S. of the answer is found by taking the complement of the right part of the multiplicand.
**Example-1 :** 436 × 99
**Step-1 :** Divide the multiplicand into two parts such that the right part has the same number of digits as the number of 9's in the multiplier.
As, there are two 9's in the multiplier, so we have two digits in the right part.
So, we have 4 in the left part and 36 in the right part.

4 | 36

**Step-2 :** L.H.S. is adding 1 to the left part and subtract it from the whole multiplicand.
Here, 4 + 1 = 5 and L.H.S. is 436 − 5 = 431

| 436 × 99 |
|---|
| 4 + 1 = 5 \| |
| 436 − 5 = 431 \| |

**Step-3 :** R.H.S is the complement of the right part of the multiplicand.
Here, the complement of 36 = 64

| 436 × 99 |
|---|
| 431 \| 100 − 36 = 64 |

∴ Answer is 43164.

**Example 2 :** 20462 × 999
**Step-1 :** Divide the multiplicand in two parts such that the right part has the same number of digits as the number 9's in the multiplier.
As, there are three 9's in the multiplier, so we have three digits in the right part.
So, we have 20 in the left part and 462 in the right part.

20 | 462

**Step-2 :** Now 20 + 1 = 21
and L.H.S. is 20462 − 21 = 20441

| 20462 × 999 |
|---|
| 20441 \| |

**Step-3 :** R.H.S. is the complement of the right part of the multiplicand. Here, the complement of 462 = 538

| 20462 × 999 |
|---|
| 20441 \| 538 |

∴ Answer is 20441538.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# TEST YOUR EXPERTISE FOR

# JEE 2020

**1.** Prove that the numbers 49, 4489, 444889,... obtained by inserting 48 into the middle of preceding number are square of integers.

**Soln. :** Given numbers are 49, 4489, 444889,...

$n^{th}$ term of the sequence is given by

$$t_n = \underbrace{4\,4\,4\,.....\,4}_{n \text{ times}}\ \underbrace{8\,8\,8\,.....\,8}_{(n-1) \text{ times}}\ 9$$

$$= 9 + (8 \times 10 + 8 \times 10^2 + ... + 8 \times 10^{n-1}) + (4 \times 10^n + 4 \times 10^{n+1} + ... + 4 \times 10^{2n-1})$$

$$= 9 + 80\left(\frac{10^{n-1}-1}{10-1}\right) + 4 \times 10^n \left(\frac{10^n-1}{10-1}\right)$$

$$= \frac{81 + 80 \times 10^{n-1} - 80 + 4 \times 10^{2n} - 4 \times 10^n}{9}$$

$$= \frac{1 + 8 \times 10^n + 4 \times 10^{2n} - 4 \times 10^n}{9} = \frac{1 + 4 \times 10^n + 4 \times 10^{2n}}{9}$$

$$= \left(\frac{2 \times 10^n + 1}{3}\right)^2, \text{ which is perfect square of an integer.}$$

**2.** Find the number of zeroes at the end of $(5^n - 1)!$.

**Soln. :** We will find exponent of 5 in $(5^n - 1)!$

$$e = \sum_{i=1}^{\infty}\left[\frac{5^n-1}{5^i}\right] = \left[\frac{5^n-1}{5}\right] + \left[\frac{5^n-1}{5^2}\right] + .... + \left[\frac{5^n-1}{5^{n-1}}\right] + 0$$

$$= \left[5^{n-1} - \frac{1}{5}\right] + \left[5^{n-2} - \frac{1}{5^2}\right] + .... + \left[5 - \frac{1}{5^{n-1}}\right]$$

$$= (5^{n-1} - 1) + (5^{n-2} - 1) + ... + (5 - 1)$$

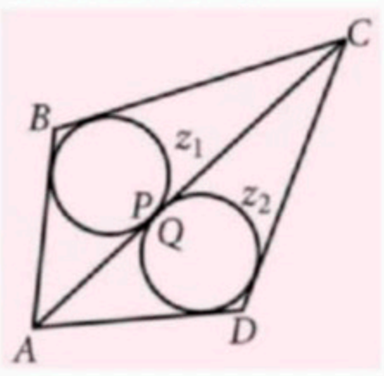
$$= (5^{n-1} + 5^{n-2} + ... + 5) - (n - 1)$$

$$= \frac{5(5^{n-1}-1)}{(5-1)} - (n-1) = \frac{5^n - 4n - 1}{4}$$

Total number of zeroes $= \dfrac{5^n - 4n - 1}{4}$

**3.** If $ABCD$ be a cyclic quadrilateral, then prove that the circles inscribed in $\Delta ABC$ and $\Delta ADC$ touch each other.

**Soln. :** Let the incircle of $\Delta ABC$ be $z_1$ and $\Delta ADC$ be $z_2$. Let $z_1$ touches $AC$ at $P$ and $z_2$ touches $AC$ at $Q$.




$\therefore \quad PQ = |AQ - AP|$

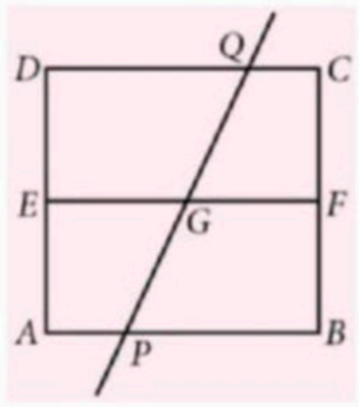
$$= \left| \frac{1}{2}(AD + AC - DC) - \frac{1}{2}(AB + AC - BC) \right|$$

$$= \left| \frac{1}{2}(AD + BC - AB - CD) \right| = 0$$

[$\because$ $ABCD$ is a cyclic quadrilateral]

**4.** Each of the given 9 lines cuts a given square into two quadrilaterals, whose areas are in the ratio 2 : 3. Prove that at least three of these lines pass through the same point.

**Soln. :** If a line cuts a square $ABCD$ into two parts it must intersect two of its sides (internally). The two parts are quadrilaterals only if it intersects a pair of opposite sides. At least five of the nine must meet the pair $AB$, $CD$ or else, the pair $AD$, $BC$. Suppose that five meet $AB$, $CD$ and let $PQ$ be one of them. Let $E$, $F$ be the mid-points of $AD$, $BC$ and $PQ$ meets $EF$ at $G$. The quadrilaterals $APQD$, $PBCQ$ formed by $PQ$ are trapeziums.




$$\therefore \quad \frac{\frac{1}{2}AD(AP + DQ)}{\frac{1}{2}BC(PB + QC)} = \frac{2}{3} \text{ or } \frac{3}{2}$$

$$\Rightarrow \quad \frac{1}{2}(AP + DQ) = EG \text{ and } \frac{1}{2}(PB + QC) = GF$$

$$\Rightarrow \quad \frac{EG}{GF} = \frac{2}{3} \text{ or } \frac{3}{2}$$

**By : Rakesh Sharma, Principal, HOD Maths, S.S. Gurukulam Academy, Jamalpur, Mirzapur**

If $G_1$ and $G_2$ are the points dividing $EF$ in the ratio $2:3$ or $3:2$, then $PQ$ contain $G_1$ or $G_2$ of the five lines meeting $AB$, $CD$ and at least three must pass through the same point $G_1$ or $G_2$.

6. *Find the value of*

$$y = 6 + \log_{\left(\frac{1}{3\sqrt{2}}\sqrt{6-\frac{1}{3\sqrt{2}}\sqrt{6-\frac{1}{3\sqrt{2}}\sqrt{6-\frac{1}{3\sqrt{2}}}}}\right)}\left[\frac{1}{3\sqrt{2}}\sqrt{6-\frac{1}{3\sqrt{2}}\sqrt{6-\frac{1}{3\sqrt{2}}\sqrt{6-\frac{1}{3\sqrt{2}}}}}\right]$$

**Solu.** : Let $x = \frac{1}{3\sqrt{2}}\sqrt{6-\frac{1}{3\sqrt{2}}\sqrt{6-\frac{1}{3\sqrt{2}}\sqrt{6-\frac{1}{3\sqrt{2}}}}}$

Then $x = \frac{1}{3\sqrt{2}}\sqrt{6-x}$

$\Rightarrow 18x^2 + x - 6 = 0 \Rightarrow (9x-4)(2x+3) = 0$

$\Rightarrow x = \frac{4}{9}, \frac{-3}{2}$

As $x > 0$, we get $x = 4/9$

Thus, $y = 6 + \log_{(x)}\left(\frac{4}{9}\right) = 6 + \log_{(x)}\left(\frac{3}{2}\right)^{-2} = 4$

7. *If* $f(x) = \frac{4^x}{4^x+2}$, *then find* $\sum_{r=1}^{2001} f\left(\frac{r}{2002}\right)$.

**Solu.** : Given, $f(x) = \frac{4^x}{4^x+2}$

$\therefore \quad f(1-x) = \frac{4^{1-x}}{4^{1-x}+2} = \frac{4}{4+2\cdot 4^x} = \frac{2}{2+4^x}$

Clearly, $f(x) + f(1-x) = 1$

Now, $\sum_{r=1}^{2001} f\left(\frac{r}{2002}\right) = f\left(\frac{1}{2002}\right) + f\left(\frac{2}{2002}\right) + f\left(\frac{3}{2002}\right) + f\left(\frac{4}{2002}\right) + \ldots + f\left(\frac{2001}{2002}\right)$

$= \left[f\left(\frac{1}{2002}\right) + f\left(\frac{2001}{2002}\right)\right] + \left[f\left(\frac{2}{2002}\right) + f\left(\frac{2000}{2002}\right)\right] + \ldots + \frac{1}{2}f\left(\frac{1000}{2002}\right) + f\left(\frac{1002}{2002}\right)\Big] + f\left(\frac{1001}{2002}\right)$

$= 1 + 1 + 1 \ldots \text{ to } 1000 \text{ terms} + f\left(\frac{1}{2}\right)$

$= 1000 + \frac{\sqrt{6}}{\sqrt{4}+2} = 1000 + \frac{1}{2} = 1000.5$

8. *Show that the polynomial* $f(x) = x^3 + px + q$ *has a repeated zero, if* $4p^3 + 27q^2 = 0$.

**Solu.** : Given, $f(x) = x^3 + px + q$ so $f'(x) = 3x^2 + p$

Zeroes of $f'(x)$ are given by $3x^2 + p = 0$

$\Rightarrow x^2 = \frac{-p}{3} \Rightarrow x = \pm\sqrt{\frac{-p}{3}}$

If $\sqrt{\frac{-p}{3}}$ is a zero of $f(x)$, then $f\left(\sqrt{\frac{-p}{3}}\right) = 0$

$\Rightarrow \left[\sqrt{\frac{-p}{3}}\right]^3 + p\left[\sqrt{\frac{-p}{3}}\right] + q = 0$

$\Rightarrow \left(\frac{-p}{3}\right)^{3/2} + p\left(\frac{-p}{3}\right)^{1/2} = -q$

Squaring both sides, we get

$\left(\frac{-p}{3}\right)^3 + p^2\left(\frac{-p}{3}\right) + 2p\left(\frac{-p}{3}\right)^{3/2}\left(\frac{-p}{3}\right)^{1/2} = q^2$

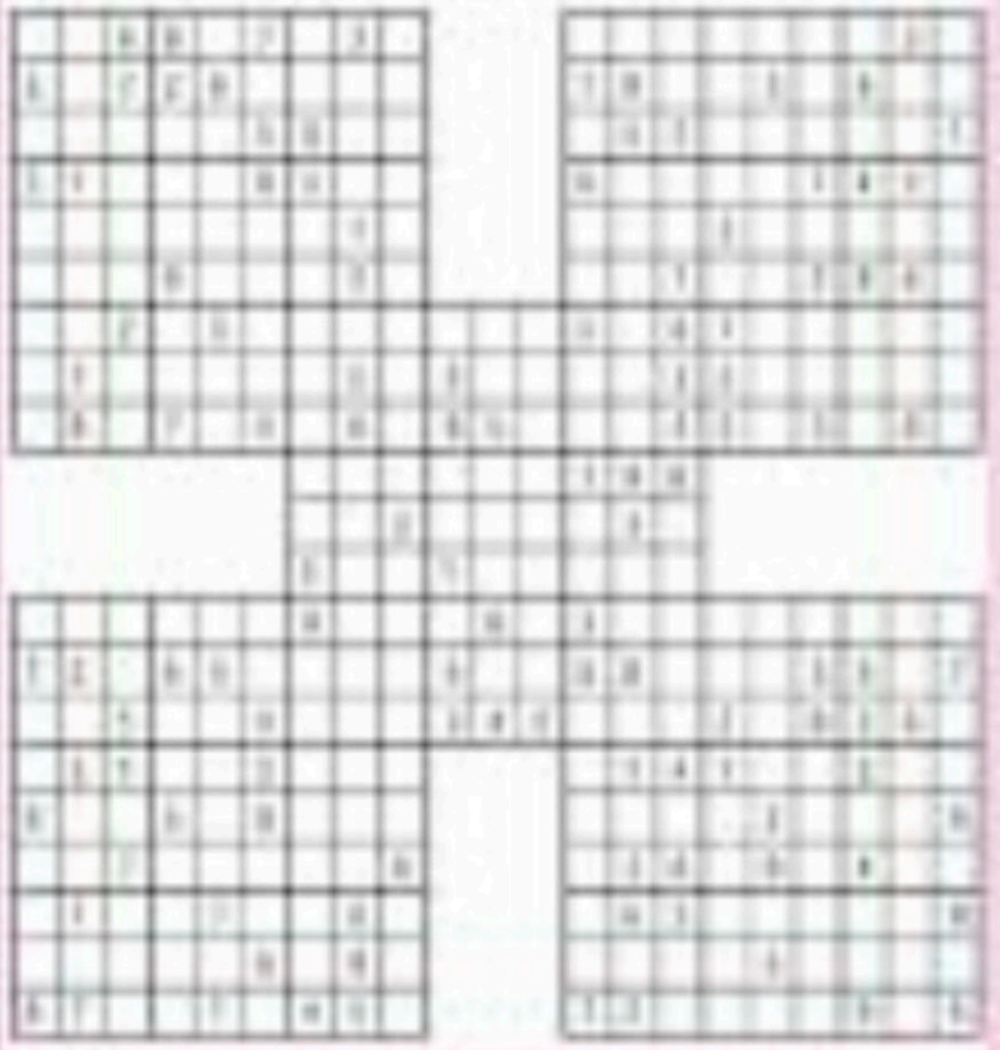
$\Rightarrow -\frac{p^3}{27} - \frac{p^3}{3} + \frac{2p^3}{9} = q^2 \Rightarrow -p^3 - 9p^3 + 6p^3 = 27q^2$

$\Rightarrow 4p^3 + 27q^2 = 0$, which is the required condition.


**SAMURAI SUDOKU**

Samurai Sudoku puzzle consists of five overlapping sudoku grids. The standard sudoku rules apply to each 9 × 9 grid. Place digits from 1 to 9 in each empty cell. Every row, every column and every 3 × 3 box should contain one of each digit.

The puzzle has a unique answer.

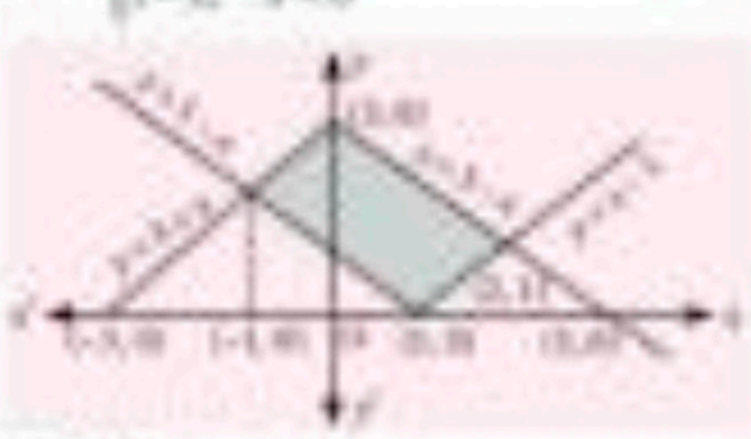


Readers can mail their responses at mtfeedback@mtg.in or post us with complete address. Winners solve with their solution feedback will be published in next issue.

8. Find the area of the region bounded by the curves $y = |x| - 3$ and $y = |x - 1|$.

**Soln.** : Since, $y = 3 - |x| = \begin{cases} 3 + x, & x < 0 \\ 3 - x, & x \geq 0 \end{cases}$

$y = |x - 1| = \begin{cases} x - 1, & x \geq 1 \\ 1 - x, & x < 1 \end{cases}$




$\therefore$ Required area

$$= \int_{-3}^{0} [(3 + x) - (1 - x)]\,dx + \int_{0}^{1} [(3 - x) - (1 - x)]\,dx + \int_{1}^{3} [(3 - x) - (x - 1)]\,dx$$

$$= \int_{-3}^{0} (2 + 2x)\,dx + \int_{0}^{1} 2\,dx + \int_{1}^{3} (4 - 2x)\,dx$$

$$= [2x + x^2]_{-3}^{0} + [2x]_{0}^{1} + [4x - x^2]_{1}^{3}$$

$= -(-3 + 9) + 2 + (12 - 9) - (4 - 1) = 1 + 3 = 4$ sq. units

9. Find the largest constant $k$ such that

$\frac{8abc}{a+b+c} \leq (a+b)^2 + (a+b+4c)^2 \ \forall\, a, b, c > 0$.

**Soln.** : By A.M.-G.M. inequality

$(a + b)^2 + (a - b + 4c)^2$

$= (a + b)^2 + [(a + b) + (-b + 2c)]^2 \geq (2\sqrt{ab})^2 + (2\sqrt{2ac} + 2\sqrt{2bc})^2$

$= 4ab + 8ac + 8bc + 16c\sqrt{ab}$

$\Rightarrow \frac{(a+b)^2 + (a+b+4c)^2}{abc} \geq \frac{4ab + 8ac + 8bc + 16c\sqrt{ab}}{abc}(a + b + c)$

$= \left(\frac{4}{c} + \frac{8}{b} + \frac{8}{a} + \frac{16}{\sqrt{ab}}\right)(a + b + c)$

$= 8\left(\frac{1}{2c} + \frac{1}{b} + \frac{1}{a} + \frac{1}{\sqrt{2ab}} + \frac{1}{\sqrt{2ab}}\right)\left(\frac{a}{2} + \frac{a}{2} + \frac{b}{2} + \frac{b}{2} + c\right)$

$\geq 8\left(5\sqrt[5]{\frac{1}{2a^2b^2c}}\right)\left(5\sqrt[5]{\frac{a^2b^2c}{2^4}}\right) = 100$

Again by A.M.-G.M. inequality.

Hence, largest constant $k$ is 100.

10. If $i = \sqrt{-1}$, then find the value of

$$4 + 3\left[\left(\frac{-1}{2}\right) + \frac{i\sqrt{3}}{2}\right]^{334} + 3\left[\left(\frac{-1}{2}\right) + \frac{i\sqrt{3}}{2}\right]^{365}$$

**Soln.** : Let $E = 4 + 3\left[\left(-\frac{1}{2}\right) + \frac{i\sqrt{3}}{2}\right]^{334} + 3\left[\left(-\frac{1}{2}\right) + \frac{i\sqrt{3}}{2}\right]^{365}$

$\because\ \omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

$\therefore\ E = 4 + 3(\omega)^{334} + 3(\omega)^{365} = 4 + 3\omega + 3\omega^2$

$= 1 + 2(\omega + \omega^2) = 1 + (-1 + i\sqrt{3}) = i\sqrt{3}$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# MATHS MUSING

## SOLUTION SET-205

**1. (c) :** $a^{12} r^{66} = 8^{2014} = 2^{6042} \Rightarrow a^2 r^{11} = 2^{1007}$

Let $a = 2^{\alpha}$, $r = 2^{\beta}$. Then, $2\alpha + 11\beta = 1007$

$\Rightarrow$ $(\alpha, \beta) = (498, 1), (487, 3), ..., (3, 91)$ *i.e.*, 46 pairs.

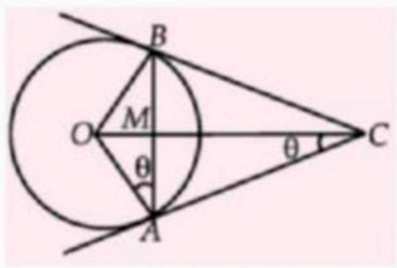
**2. (b) :** $M = (2, 3)$, $OM = \sqrt{13}$ and $OA = 5$

$\Rightarrow$ $MA = \sqrt{25 - 13} = 2\sqrt{3}$

$\therefore$ Area of $OACB = 2\,\Delta OAC$

$= OA \cdot AC = 5 \cdot 5\cot\theta$

$= 25 \cdot \dfrac{2\sqrt{3}}{\sqrt{13}} = 50\sqrt{\dfrac{3}{13}}$




**3. (a) :** The number of all 5 digit numbers is $6! - 5! = 600$

Numbers divisible by 11 without digit 1 are

23045 with 8 permutations

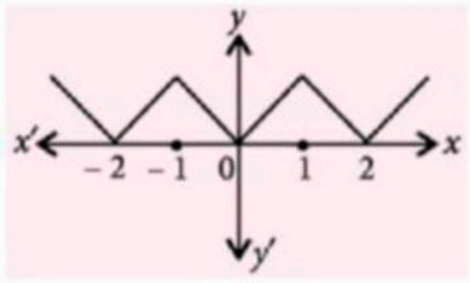
32450 with 8 permutations

Likewise there are 16 numbers without 3 and 16 numbers without 5

$\therefore$ The number of numbers divisible by 11 are $16 \times 3 = 48$

$$\text{Required probability} = \frac{48}{600} = \frac{2}{25}$$

**4. (d) :**




$f(x) = \left|1 - \left|1 - |x|\right|\right|$ is not differentiable at $x = 0, \pm 1, \pm 2$

**5. (c) :** $\displaystyle\int_0^{100} \{\sqrt{x}\}dx = \int_0^{100} \sqrt{x}dx - \sum_{r=0}^{9} \int_{r^2}^{(r+1)^2} r dx$

$$= \frac{2000}{3} - \sum_{r=1}^{9}(2r^2 + r) = \frac{2000}{3} - 615 = 51\frac{2}{3}$$

**6. (a, b, c) :** $\displaystyle S = \sum_{r=0}^{5}(z_1 + z_2\alpha^r)(\overline{z}_1 + \overline{z}_2\overline{\alpha^r})$

$$= \sum_{r=0}^{5}(z_1 + z_2\alpha^r)(\overline{z}_1 + \overline{z}_2\alpha^{6-r})$$

$$= \sum_{r=0}^{5}\left[|z_1|^2 + |z_2|^2 + \overline{z}_1 z_2\alpha^r + z_1\overline{z}_2\alpha^{6-r}\right]$$

$$= \sum_{r=0}^{5}\left(|z_1|^2 + |z_2|^2\right) \qquad \left[\because \sum_{r=0}^{5}\alpha^r = 0\right]$$

$= 6(25 + 5) = 180 = 2^2 \cdot 3^2 \cdot 5$

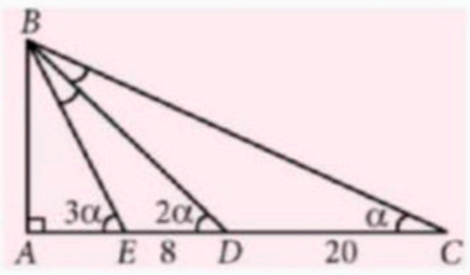
**7. (b) :** $\angle DBC = \alpha \Rightarrow BD = 20$, $\angle EBD = \alpha$

By sine rule for $\Delta EBD$, we get

$$\frac{\sin 3\alpha}{20} = \frac{\sin\alpha}{8}$$

$$\Rightarrow 3 - 4\sin^2\alpha = \frac{5}{2}$$

$$\Rightarrow \sin^2\alpha = \frac{1}{8}, \quad \cos 2\alpha = \frac{3}{4}.$$




**8. (d) :** $AE + 28 = AB\cot\alpha$ and $AE + 8 = AB\cot 2\alpha$

On subtracting, we get $20 = AB(\cot\alpha - \cot 2\alpha)$

$$\therefore \quad AB = 20\sin 2\alpha = 20\sqrt{1 - \frac{9}{16}} = 5\sqrt{7}$$

$$\therefore \quad AE + 8 = AB\cot 2\alpha = 5\sqrt{7} \cdot \frac{3}{\sqrt{7}} = 15 \Rightarrow AE = 7$$

**9. (3) :** $\vec{a} \cdot (\vec{a} \times \vec{r} + \vec{b}) = \vec{a} \cdot \vec{r} \Rightarrow \vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{r} = 2$

Also, $\vec{a} \times (\vec{a} \times \vec{r} + \vec{b}) = \vec{a} \times \vec{r}$

$\Rightarrow$ $(\vec{a} \cdot \vec{r})\vec{a} - (\vec{a} \cdot \vec{a})\vec{r} + \vec{a} \times \vec{b} = \vec{r} - \vec{b}$

$$\therefore \quad \vec{r} = \frac{1}{2}\left(2\vec{a} + \vec{b} + \vec{a} \times \vec{b}\right)$$

The projection of $\vec{r}$ along $\vec{b}$

$$= \vec{r} \cdot \frac{\vec{b}}{|\vec{b}|} = \frac{1}{2|\vec{b}|}\left(2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}\right) = \frac{4 + 16}{2 \times 4} = \frac{5}{2}$$

$\Rightarrow$ $m = 5, n = 2 \Rightarrow m - n = 3$

**10. (b) : P $\rightarrow$ 4; Q $\rightarrow$ 1; R $\rightarrow$ 3; S $\rightarrow$ 2**

**(P)** The desired number is the coefficient of $x^{10}$ in $(x + x^2 + ... + x^6)^4$ or coefficient of $x^6$ in $(1 + x + ... + x^5)^4$

$= (1 - x^6)^4 (1 - x)^{-4}$

$$= (1 - 4x^6 + ...)\left(1 + \binom{4}{1}x + \binom{5}{2}x^2 + ...\right)$$

$$\therefore \quad \text{Required coefficient} = \binom{9}{6} - 4 = 84 - 4 = 80$$

**(Q)** $a = 2\alpha - 1$, $b = 2\beta - 1$, $c = 2\gamma - 1$, $d = 2\delta - 1$

$\Rightarrow$ $\alpha + \beta + \gamma + \delta = 10$

$\therefore$ The number of solutions is $\binom{9}{3} = 84$

**(R)** $\left(1 + (x + 2y)\right)^5 = 1 + \binom{5}{1}(x + 2y) + \binom{5}{2}(x + 2y)^2$

$$+ \binom{5}{3}(x + 2y)^3 + .... + \binom{5}{5}(x + 2y)^5$$

$$\therefore \quad \text{The coefficient of } x^2 y = \binom{5}{3} \cdot 3 \cdot 2 = 60$$

**(S)** $\displaystyle\sum_{r=1}^{10}\frac{r \cdot C_r}{C_{r-1}} = \sum_{r=1}^{10}\frac{r(10 - r + 1)}{r} = \sum_{r=1}^{10}(11 - r)$

$$= 10 + 9 + 8 + ... + 1 = \frac{10 \cdot 11}{2} = 55$$

# CONCEPT BOOSTER

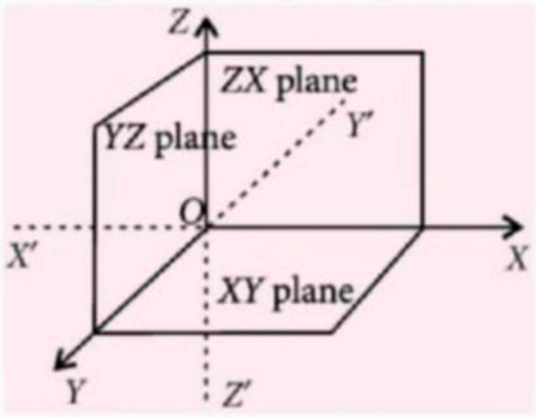
## Three Dimensional Geometry

Class XII

*ALOK KUMAR

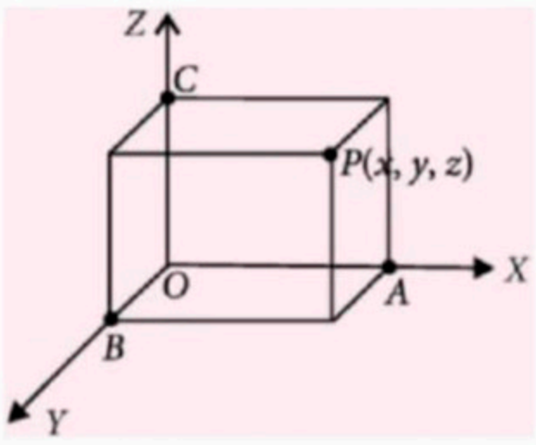
### RECTANGULAR COORDINATE SYSTEM IN SPACE

- Let '$O$' be any point in space and $\overleftrightarrow{X'OX}$, $\overleftrightarrow{Y'OY}$ and $\overleftrightarrow{Z'OZ}$ be three lines perpendicular to each other. These lines are known as coordinate axes and $O$ is called origin. The planes $XY$, $YZ$, $ZX$ are known as the coordinate planes.




#### Coordinates of a Point in Space

- Consider a point $P$ in space. The position of the point $P$ is given by triad $(x, y, z)$, where $x$, $y$ and $z$ are perpendicular distance from $YZ$-plane, $ZX$-plane and $XY$-plane respectively.

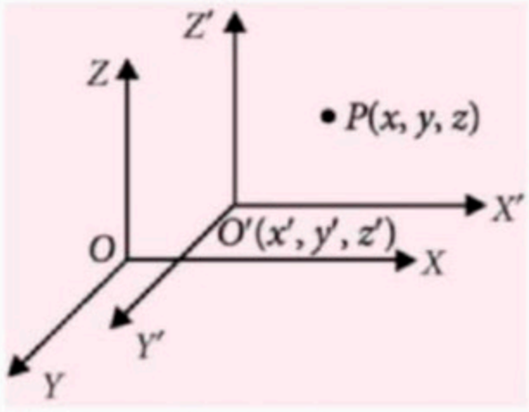



- If we assume $\hat{i}, \hat{j}, \hat{k}$ unit vectors along $OX$, $OY$, $OZ$ respectively, then position vector of point $P$ is $x\hat{i} + y\hat{j} + z\hat{k}$ or simply $(x, y, z)$.

#### Translation of Axes

Shifting the origin to another point without changing the directions of the axes is called the translation of axes.

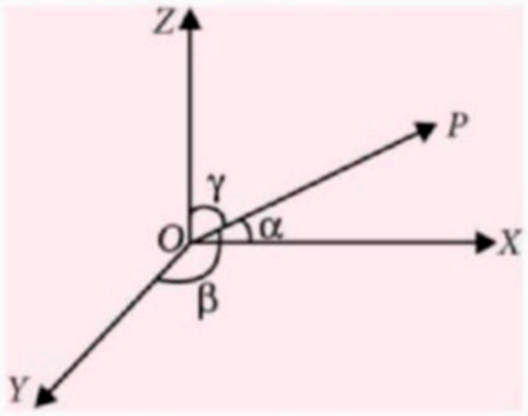
- Let the origin $O$ be shifted to another point $O'(x', y', z')$ without changing the direction of axes. Let the new coordinate frame be $O'X'Y'Z'$. Also, let $P(x, y, z)$ be a point with respect to the coordinate frame $OXYZ$.




Then, coordinates of point $P$ w.r.t. new coordinate frame $O'X'Y'Z'$ are $(x_1, y_1, z_1)$, where $x_1 = x - x'$, $y_1 = y - y'$, $z_1 = z - z'$

### DIRECTION COSINES OF A LINE

- If $\alpha, \beta, \gamma$ be the angles which a given directed line makes with the positive directions of the coordinate axes, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of the given line and are generally denoted by $l, m, n$ respectively.
Thus, $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.




- Direction cosines of the $x$-axis are $\cos0°$, $\cos90°$, $\cos 90°$ *i.e.*, (1, 0, 0) and direction cosines of the $y$- axis and $z$- axis are respectively (0, 1, 0) and (0, 0, 1).

#### Relation Between the Direction Cosines

- Let $OP$ be any line having direction cosines $l, m, n$ then, $l^2 + m^2 + n^2 = 1$.

#### Direction Ratios

- If $a, b, c$ are three numbers proportional to the direction cosines $l, m, n$ then $a, b, c$ are called its direction ratios. They are also called direction numbers or direction components.

---

*Alok Kumar, a B.Tech from IIT Kanpur and INMO 4th ranker of his time, has been training IIT and Olympiad aspirants for close to two decades now. His students have bagged AIR 1 in IIT JEE and also won medals for the country at IMO. He has also taught at Maths Olympiad programme at Cornell University, USA and UT, Dallas. He has been regularly proposing problems in international Mathematics journals.

By definition, we have

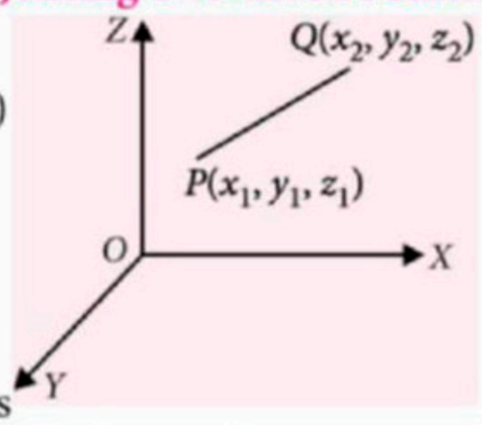
$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$$

$$\Rightarrow\ l=\pm\frac{a}{\sqrt{\sum a^2}}, m=\pm\frac{b}{\sqrt{\sum a^2}}$$

and $n=\pm\frac{c}{\sqrt{\sum a^2}}$, where the same sign either positive or negative is to be chosen throughout.

## Direction Cosines of a Line Joining Two Given Points

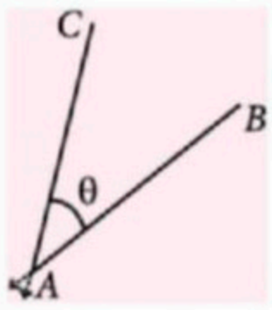
- The direction ratios of line $PQ$ joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1 = a$(say), $y_2 - y_1 = b$ (say) and $z_2 - z_1 = c$ (say). Then direction cosines are $l=\frac{(x_2-x_1)}{\sqrt{\Sigma(x_2-x_1)^2}}, m=\frac{(y_2-y_1)}{\sqrt{\Sigma(x_2-x_1)^2}}$,




$$n=\frac{(z_2-z_1)}{\sqrt{\Sigma(x_2-x_1)^2}}$$

## Angle Between Two Lines

- Let θ be the angle between two straight lines $AB$ and $AC$ whose direction cosines are $l_1, m_1, n_1$ and $l_2, m_2, n_2$ respectively, then θ is given by $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$.




- If direction ratios of two lines are $a_1, b_1, c_1$ and $a_2, b_2, c_2$ are given, then angle between two lines is given by

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}$$

## Condition of Perpendicularity

- If the given lines are perpendicular, then $\theta = 90°$ *i.e.*, $\cos\theta = 0$

$\Rightarrow l_1l_2 + m_1m_2 + n_1n_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

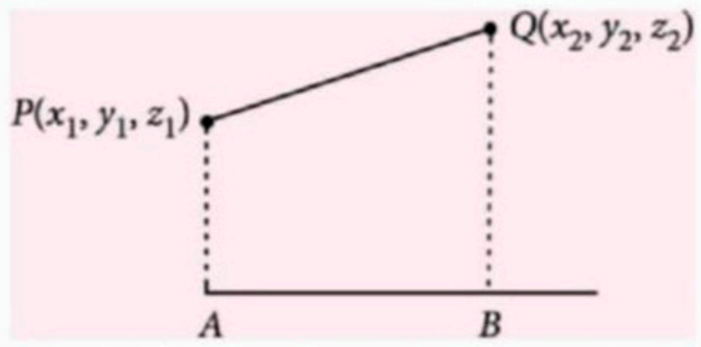
## Condition of Parallelism

- If the given lines are parallel, then $\theta = 0°$ *i.e.*, $\sin\theta = 0$

$$\Rightarrow\ \frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}\cdot\ \text{Similarly,}\ \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}\cdot$$

## Projection of a Line

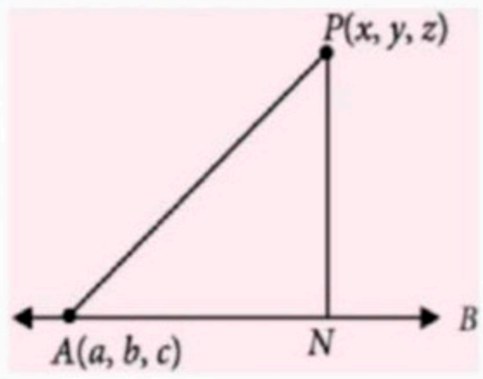
- Projection of the line joining two point $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line whose direction cosines are $l, m, n$ is




$$AB = |l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$$

## Perpendicular Distance of a Point from a Line

- Let $AB$ is a straight line passing through point $A(a, b, c)$ and having direction cosines $l, m, n$. $AN$ = projection of line $AP$ on straight line $AB$ $= l(x - a) + m(y - b) + n(z - c)$




and $AP=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$

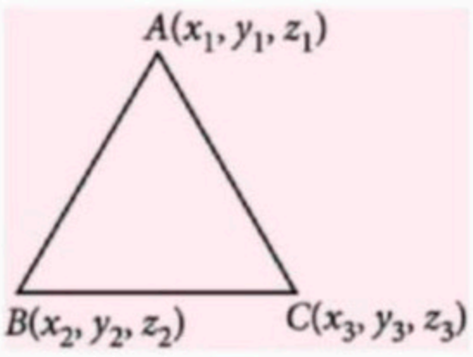
$\therefore$ Perpendicular distance of point $P$ from line $AB$ is $PN=\sqrt{AP^2-AN^2}$

## Area of a Triangle in 3D

$$\Delta_x=\frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1\\ y_2 & z_2 & 1\\ y_3 & z_3 & 1\end{vmatrix}$$

$$\Delta_y=\frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1\\ x_2 & z_2 & 1\\ x_3 & z_3 & 1\end{vmatrix}$$

$$\Delta_z=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$$




So, area of $\Delta ABC$ is given by the relation

$$\Delta^2=\Delta_x^2+\Delta_y^2+\Delta_z^2$$

# THE PLANE

## Equation of Plane in Different Forms

- General equation of a plane is $ax + by + cz + d = 0$.
- Equation of the plane in normal form is $lx + my + nz = p$, where $p$ is the length of the normal from the origin to the plane and $(l, m, n)$ be the direction cosines of the normal.
- The equation of the plane passing through $P(x_1, y_1, z_1)$ and having direction ratios $(a, b, c)$ for its normal is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

- The equation of the plane passing through three non-collinear points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ and $(x_3, y_3, z_3)$ is

$$\begin{vmatrix} (x-x_1) & (y-y_1) & (z-z_1) \\ (x_2-x_1) & (y_2-y_1) & (z_2-z_1) \\ (x_3-x_1) & (y_3-y_1) & (z_3-z_1) \end{vmatrix} = 0$$

- The equation of the plane whose intercepts are $a, b, c$ on the $x, y, z$ axes respectively is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \ (a, b, c \neq 0)$$

- Equation of $YZ$ plane is $x = 0$, equation of plane parallel to $YZ$ plane is $x = d$.
  Equation of $ZX$ plane is $y = 0$, equation of plane parallel to $ZX$ plane is $y = d$.
  Equation of $XY$ plane is $z = 0$, equation of plane parallel to $XY$ plane is $z = d$.
- Four points namely $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ will be coplanar if one point lies on the plane passing through other three points.

### Angle Between the Planes

- Angle between the planes is defined as angle between normals of the planes drawn from any point to the planes.
- Angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

$$\cos^{-1}\left(\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)\left(a_2^2+b_2^2+c_2^2\right)}}\right)$$

- If $a_1a_2 + b_1b_2 + c_1c_2 = 0$, then the planes are perpendicular to each other.
- If $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$, then the planes are parallel to each other.

### Perpendicular Distance

- The length of the perpendicular from the point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$$

### Family of planes

- Equation of plane passing through the line of intersection of two planes $u = 0$ and $v = 0$ is $u + \lambda v = 0$.

### Intersection of a Line and Plane

- If equation of a plane is $ax + by + cz + d = 0$, then direction ratios of normal to this plane are $a, b, c$.

So, angle between normal to the plane and a straight line having direction cosines $l, m, n$ is given by $\cos\theta=\left|\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}}\right|$, then angle between the plane and the straight line is $\frac{\pi}{2}-\theta$.

- Plane and straight line will be parallel if $al + bm + cn = 0$.
- Plane and straight line will be perpendicular if $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$.

### Bisector Planes of Angle Between Two Planes

- The equation of the planes bisecting the angles between two given planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

## THE STRAIGHT LINE

- Straight line in three dimensional geometry is defined as intersection of two planes. So, general equation of straight line is stated as the equations of two plane together *i.e.* general equation of straight line is $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$.

### Equation of Straight line in Different Forms

- **Symmetrical form**
  Equation of straight line passing through point $P(x_1, y_1, z_1)$ and whose direction cosines are $l, m, n$ is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

- Equation of straight line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

- The coordinates of any point on a line is given by $(x_1 + lr, y_1 + mr, z_1 + nr)$ where $r$ is any real number and $(x_1, y_1, z_1)$ is the point whose coordinates is to be given.

### Shortest Distance Between two Non-intersecting Lines

- Two lines are called non intersecting lines if they do not lie in the same plane. The straight line which is perpendicular to each of non-intersecting lines is called the line of shortest distance.

- Let the equation of two non-intersecting lines be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}=r_1 \text{ (say)} \quad \text{... (i)}$$

and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}=r_2$ (say) ... (ii)

Any point on line (i) is

$P(x_1 + l_1r_1, y_1 + m_1r_1, z_1 + n_1r_1)$

and on line (ii) is

$Q(x_2 + l_2r_2, y_2 + m_2r_2, z_2 + n_2r_2)$.

- Let $PQ$ be the line of shortest distance. Its direction ratios will be

$[(l_1r_1 + x_1 - x_2 - l_2r_2), (m_1r_1 + y_1 - y_2 - m_2r_2), (n_1r_1 + z_1 - z_2 - n_2r_2)]$

This line is perpendicular to both given line. By using condition of perpendicularity we obtain two equations in $r_1$ and $r_2$.

So by solving these, values of $r_1$ and $r_2$ can be found, and subsequently point $P$ and $Q$ can be found. The distance $PQ$ is shortest distance.

$\therefore$ The shortest distance can be found by

$$PQ=\left|\frac{\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\sum(l_1m_2-l_2m_1)^2}}\right|$$

- If any straight line is given in general form, then it can be transformed into symmetrical form and then we can further proceed.

## THE SPHERE

- A sphere is a locus of a point which moves in space such that its distance from a fixed point is constant. Fixed point is called centre of sphere and constant distance is called radius of sphere.

**Equation of Sphere in Different Forms**

- If centre of sphere is $(a, b, c)$ and radius is $r$, then equation of sphere is $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$.
- If centre of sphere is origin and radius is $r$, then $x^2 + y^2 + z^2 = r^2$.
- **General form :** The general equation of a sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, where centre of sphere $= (-u, -v, -w)$, and radius $= \sqrt{u^2+v^2+w^2-d}$
- **Diameter form:** Equation of a sphere whose extremities of diameter are $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

## PROBLEMS

### Single Correct Answer Type

**1.** The angle between two lines whose direction cosines are given by the equations $l + m + n = 0$, $l^2 + m^2 + n^2 = 0$ is

(a) $\pi/3$ (b) $\pi/2$
(c) $\pi/4$ (d) none of these

**2.** The equation of the plane which contains the line of intersection of the planes $x + y + z - 6 = 0$ and $2x + 3y + z + 5 = 0$ and perpendicular to the $xy$ plane is

(a) $x - 2y + 11 = 0$ (b) $x + 2y + 11 = 0$
(c) $x + 2y - 11 = 0$ (d) $x - 2y - 11 = 0$

**3.** The coordinates of the foot of the perpendicular drawn from the origin to the plane $3x + 4y - 6z + 1 = 0$ are

(a) $\left(\frac{-3}{61}, \frac{4}{61}, \frac{6}{61}\right)$ (b) $\left(\frac{3}{61}, \frac{-4}{61}, \frac{6}{61}\right)$
(c) $\left(\frac{-3}{61}, \frac{-4}{61}, \frac{6}{61}\right)$ (d) $\left(\frac{3}{61}, \frac{4}{61}, \frac{6}{61}\right)$

**4.** The image of the point (1, 3, 4) in the plane $2x - y + z + 3 = 0$ is

(a) (–3, 5, 2) (b) (3, 2, 5)
(c) (–5, 3, –2) (d) (–2, 5, 3)

**5.** The equation of the sphere which passes through the points (1, 0, 0), (0, 1, 0) and (0, 0, 1) and has its radius as small as possible is

(a) $3(x^2 + y^2 + z^2) + 2(x + y + z) - 1 = 0$
(b) $3(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0$
(c) $(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0$
(d) $(x^2 + y^2 + z^2) + 2(x + y + z) - 1 = 0$

**6.** A point moves so that the ratio of its distances from two fixed points is constant but not 1. Its locus is a

(a) plane (b) straight line
(c) circle (d) sphere

**7.** A variable plane passes through a fixed point $(a, b, c)$ and meets the coordinate axes in $A$, $B$, $C$. The locus of the point common to plane through $A$, $B$, $C$ parallel to coordinate planes is

(a) $ayz + bzx + cxy = xyz$
(b) $axy + byz + czx = xyz$
(c) $axy + byz + czx = abc$
(d) $bcx + acy + abz = abc$

**8.** A straight line passes through the point (2, –1, –1). It is parallel to the plane $4x + y + z + 2 = 0$ and is perpendicular to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z-5}{1}$. The equation of the straight line is

(a) $\frac{x-2}{4}=\frac{y+1}{1}=\frac{z+1}{1}$ (b) $\frac{x+2}{4}=\frac{y-1}{1}=\frac{z-1}{1}$

(c) $\frac{x-2}{-1}=\frac{y+1}{1}=\frac{z+1}{3}$ (d) $\frac{x+2}{-1}=\frac{y-1}{1}=\frac{z-1}{3}$

**9.** The direction cosines of the line which is perpendicular to the lines with direction cosines proportional to (1, –2, –2), (0, 2, 1) is

(a) $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ (b) $\left(\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}\right)$

(c) $\left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$ (d) $\left(\frac{-2}{3}, \frac{-1}{3}, \frac{-2}{3}\right)$

**10.** The points (4, 7, 8), (2, 3, 4), (–1, –2, 1) and (1, 2, 5) are

(a) the vertices of a parallelogram
(b) collinear
(c) the vertices of a trapezium
(d) concyclic

**11.** A plane is passed through the middle point of the line segment joining $A(-2, 5, 1)$, $B(6, 1, 5)$ and is perpendicular to this line. Its equation is

(a) $2x - y - z = 4$ (b) $2x - y + z = 4$

(c) $x - 3y + z = 5$ (d) $x - 4y + 2z = 5$

**12.** The angle between straight lines whose direction cosines are $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ is

(a) $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (b) $\cos^{-1}\left(\frac{1}{\sqrt{6}}\right)$

(c) $\cos^{-1}\left(-\frac{3}{\sqrt{6}}\right)$ (d) none of these

**13.** Which one of the following is best condition for the plane $ax + by + cz + d = 0$ to intersect the $x$ and $y$ axes at equal angle?

(a) $|a| = |b|$ (b) $a = -b$

(c) $a = b$ (d) $a^2 + b^2 = 1$

**14.** A straight line is inclined to the axes of $x$ and $z$ at an angels 45° and 60° respectively, then the inclination of the line to the $y$-axis is

(a) 30° (b) 45° (c) 60° (d) 90°

**15.** $A$ (3, 7, 5) and $B$ (–3, 2, 6) be the two points. The projection of $AB$ on the line which joins the points (7, 9, 4) and (4, 5, –8) is

(a) 26 (b) 2 (c) 13 (d) 4

**16.** The point of intersection of the $xy$-plane and the line passing through the points $A \equiv (3, 4, 1)$ and $B \equiv (5, 1, 6)$ is

(a) $\left(-\frac{13}{5}, \frac{23}{5}, 0\right)$ (b) $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$

(c) $\left(\frac{13}{5}, -\frac{23}{5}, 0\right)$ (d) $\left(-\frac{13}{5}, -\frac{23}{5}, 0\right)$

**17.** The direction cosines of two mutually perpendicular lines are $l_1$, $m_1$, $n_1$ and $l_2$, $m_2$, $n_2$. The direction cosines of the line perpendicular to both the given lines will be

(a) $l_1 + l_2, m_1 + m_2, n_1 + n_2$
(b) $l_1 - l_2, m_1 - m_2, n_1 - n_2$
(c) $l_1l_2, m_1m_2, n_1n_2$
(d) $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$

**18.** The plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(a)$ with $x$-axis. The value of '$a$' is equal to

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{2}}{3}$ (c) $\frac{2}{7}$ (d) $\frac{3}{7}$

**19.** The direction ratios of a normal to the plane passing through (1, 0, 0), (0, 1, 0) and making an angle $\pi/4$ with the plane $x + y = 3$ are

(a) $(1, \sqrt{2}, 1)$ (b) $(1, 1, \sqrt{2})$

(c) (1, 1, 2) (d) $(\sqrt{2}, 1, 1)$

**20.** The equation of a plane passing through the line of intersection of the planes $x + y + z = 5$, $2x - y + 3z = 1$ and parallel to the line $y = z = 0$ is

(a) $3x - z = 9$ (b) $3y - z = 9$

(c) $x - 3z = 9$ (d) $y - 3z = 9$

**21.** The equation of the plane passing through the points (2, –1, 0), (3, –4, 5) and parallel to the line $2x = 3y = 4z$ is

(a) $125x - 90y - 79z = 340$
(b) $32x - 21y - 36z = 85$
(c) $73x + 61y - 22z = 85$
(d) $29x - 27y - 22z = 85$

**22.** The straight lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$, will intersect provided

(a) $k \in \{3, -3\}$ (b) $k \in \{0, -1\}$

(c) $k \in \{-1, 1\}$ (d) $k \in \{0, -3\}$

## Multiple Correct Answer Type

**23.** The line $\frac{x-2}{3}=\frac{y-1}{2}=\frac{z-1}{-1}$ intersects the curve $x^2 - y^2 = a^2; z = 0$, if $a$ is equal to

(a) 4 (b) $\sqrt{5}$ (c) –4 (d) 3

**24.** If $P_1$, $P_2$, $P_3$ denote the distances of the plane $2x - 3y + 4z + 2 = 0$ from the planes $2x - 3y + 4z + 6 = 0$, $4x - 6y + 8z + 3 = 0$ and $2x - 3y + 4z - 6 = 0$ respectively, then

(a) $P_1 + 8P_2 - P_3 = 0$ (b) $P_3 = 16P_2$

(c) $8P_2 = P_1$ (d) $P_1 + 2P_2 + 3P_3 = \sqrt{29}$

**25.** The plane $3x + 4y = 0$ is rotated through an angle of $\pi/4$ about its line of intersection with the $xy$-plane. The equation of the plane in the new position is

(a) $3x + 4y - z = 0$ (b) $3x + 4y + z = 0$

(c) $3x + 4y + 5z = 0$ (d) $3x + 4y - 5z = 0$

**26.** If the median through $A$ of $\Delta ABC$ having vertices $A \equiv (2, 3, 5)$, $B \equiv (-1, 3, 2)$ and $C \equiv (\lambda, 5, \mu)$ is equally inclined to the axes, then

(a) $\lambda = 7$ (b) $\mu = 10$ (c) $\lambda = 10$ (d) $\mu = 7$

## Comprehension Type

### Paragraph for Q. No. 27 to 29

$(a_1a_2 + b_1b_2 + c_1c_2)^2 \le (a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)$ and equality holds, when $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

**27.** If $(l, m, n)$ are direction cosines of a line, then the range of values of $3l + 4m - 5n$ is

(a) $[-5\sqrt{2}, 5\sqrt{2}]$ (b) $[-6\sqrt{2}, 6\sqrt{2}]$

(c) $[-7\sqrt{2}, 7\sqrt{2}]$ (d) $[-8\sqrt{2}, 8\sqrt{2}]$

**28.** If $3l + 4m - 5n$ takes its maximum value, then the value of $|lm + mn + nl|$ is equal to

(a) 0.12 (b) 0.46 (c) 1.08 (d) 1.72

**29.** If $ax + by + cz = \sqrt{a+b+c}$ ($a, b, c$ are fixed +ve real numbers), then the minimum value of $ax^2 + by^2 + cz^2$ is

(a) 1 (b) $a + b + c$

(c) $a^2 + b^2 + c^2$ (d) $(a + b + c)^2$

### Paragraph for Q. No. 30 to 32

Let $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ be two planes, where $d_1, d_2 > 0$. Then origin lies in acute angle if $a_1a_2 + b_1b_2 + c_1c_2 < 0$ and origin lies in obtuse angle if $a_1a_2 + b_1b_2 + c_1c_2 > 0$.

Further point $(x_1, y_1, z_1)$ and origin both lie either in acute angle or in obtuse angle, if $(a_1x_1 + b_1y_1 + c_1z_1 + d_1)(a_2x_1 + b_2y_1 + c_2z_1 + d_2) > 0$.

One of $(x_1, y_1, z_1)$ and origin lie in acute angle and the other in obtuse angle, if $(a_1x_1 + b_1y_1 + c_1z_1 + d_1)(a_2x_1 + b_2y_1 + c_2z_1 + d_2) < 0$.

**30.** Given the planes $2x + 3y - 4z + 7 = 0$ and $x - 2y + 3z - 5 = 0$, if a point $P$ is $(1, -2, 3)$, then

(a) $O$ and $P$ both lie in acute angle

(b) $O$ and $P$ both lie in obtuse angle

(c) $O$ lies in acute angle, $P$ lies in obtuse angle

(d) $O$ lies in obtuse angle, $P$ lies in acute angle

**31.** Given the planes $x + 2y - 3z + 5 = 0$ and $2x + y + 3z + 1 = 0$. If a point $P$ is $(2, -1, 2)$, then

(a) $O$ and $P$ both lie in acute angle

(b) $O$ and $P$ both lie in obtuse angle

(c) $O$ lies in acute angle, $P$ lies in obtuse angle

(d) $O$ lies in obtuse angle, $P$ lies in acute angle

**32.** Given the planes $x + 2y - 3z + 2 = 0$ and $x - 2y + 3z + 7 = 0$, if the point $P$ is $(1, 2, 2)$, then

(a) $O$ and $P$ both lie in acute angle

(b) $O$ and $P$ both lie in obtuse angle

(c) $O$ lies in acute angle, $P$ lies in obtuse angle

(d) $O$ lies in obtuse angle, P lies in acute angle.

## Matrix-Match Type

**33.** Match the following :

| Column I | | Column II | |
|---|---|---|---|
| A. | $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-1}{3} = \frac{y-3}{4} = \frac{z-5}{5}$ are | p. | Coincident lines |
| B. | $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-3}{2} = \frac{y-5}{3} = \frac{z-7}{4}$ are | q. | Parallel and different lines |
| C. | $\frac{x-2}{5} = \frac{y+3}{4} = \frac{z-5}{-2}$ and $\frac{x-7}{5} = \frac{y-1}{4} = \frac{z-2}{-2}$ are | r. | Skew lines |
| D. | $\frac{x-3}{2} = \frac{y+2}{3} = \frac{z-4}{5}$ and $\frac{x-3}{3} = \frac{y-2}{2} = \frac{z-7}{5}$ are | s. | Intersecting in a point |
| | | t. | Coplanar lines |

**34.** Match the following :

| Column I | | Column II | |
|---|---|---|---|
| A. | Coordinates of foot of perpendicular drawn for point (1, 2, 3) to the line $\frac{x-2}{2} = \frac{y-1}{3} = \frac{z-2}{4}$, is | p. | $\left(\frac{107}{29}, \frac{30}{29}, \frac{69}{29}\right)$ |

| | | | |
|---|---|---|---|
| B. | Coordinates of image of the point (1, 2, 3) in the line $\frac{x-2}{2}=\frac{y-1}{3}=\frac{z-2}{4}$, is | q. | $\left(\frac{88}{29},\frac{125}{29},\frac{69}{29}\right)$ |
| C. | Coordinates of foot of perpendicular from the point (2, 3, 5) to the plane $2x + 3y - 4z + 17 = 0$, is | r. | $\left(\frac{38}{29},\frac{57}{29},\frac{185}{29}\right)$ |
| D. | Coordinates of image of the point (2, 5, 1) in the plane $3x - 2y + 4z - 5 = 0$, is | s. | $\left(\frac{68}{29},\frac{44}{29},\frac{78}{29}\right)$ |

**35.** A variable plane cuts the $x$-axis, $y$-axis and $z$-axis at the points $A$, $B$ and $C$ respectively such that the volume of the tetrahedron $OABC$ remains constant equal to 32 cubic unit and $O$ is the origin of the coordinate system, then match the following.

| Column – I | | Column – II | |
|---|---|---|---|
| A. | The locus of the centroid of the tetrahedron is | p. | $xyz = 24$ |
| B. | The locus of the point equidistant from $O$, $A$, $B$ and $C$ is | q. | $(x^2 + y^2 + z^2)^3 = 192\ xyz$ |
| C. | The length of the foot of perpendicular from origin to the plane is | r. | $xyz = 3$ |
| D. | If $PA$, $PB$ and $PC$ are mutually perpendicular, then the locus of $P$ is | s. | $(x^2 + y^2 + z^2)^3 = 1536\ xyz$ |

## SOLUTIONS

**1. (a) :** Eliminating $n$ between the two relations, we have $l^2 + m^2 - (l + m)^2 = 0 \Rightarrow 2lm = 0$

$\Rightarrow$ either $l = 0$ or $m = 0$

If $l = 0$, then $m + n = 0$ *i.e.*, $m = -n$

$\Rightarrow \frac{l}{0}=\frac{m}{1}=\frac{n}{-1}$, giving the direction ratios of one line.

If $m = 0$, then $l + n = 0$ *i.e.*, $l = -n$

$\Rightarrow \frac{l}{1}=\frac{m}{0}=\frac{n}{-1}$, giving the direction ratios of other line.

The angles between these lines is

$$\cos^{-1}\left|\frac{0\cdot1+1\cdot0+(-1)(-1)}{\sqrt{0^2+1^2+(-1)^2}\ \sqrt{1^2+0^2+(-1)^2}}\right|=\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$$

**2. (b) :** Equation of the required plane is

$(x + y + z - 6) + \lambda(2x + 3y + z + 5) = 0$

$\Rightarrow (1 + 2\lambda)x + (1 + 3\lambda)y + (1 + \lambda)z + (-6 + 5\lambda) = 0$

This plane is perpendicular to $xy$ plane whose equation is $z = 0$ *i.e.*, $0 \cdot x + 0 \cdot y + z = 0$

$\therefore$ By condition of perpendicularity

$0\cdot(1 + 2\lambda) + 0\cdot(1 + 3\lambda) + (1 + \lambda)\cdot1 = 0 \Rightarrow \lambda = -1$

$\therefore$ Equation of required plane is

$(1 - 2)x + (1 - 3)y + (1 - 1)z + (-6 - 5) = 0$

$\Rightarrow x + 2y + 11 = 0$

**3. (c) :** The equation of the plane is

$3x + 4y - 6z + 1 = 0$ ...(i)

The direction ratios of the normal to the plane (i) are 3, 4, –6. So, equation of the line through (0, 0, 0) and perpendicular to the plane (i) are

$$\frac{x}{3}=\frac{y}{4}=\frac{z}{-6}=r \text{ (say)} \quad \text{...(ii)}$$

The coordinates of any point $P$ on (ii) are $(3r, 4r, -6r)$. If this point lie on the plane (i), then

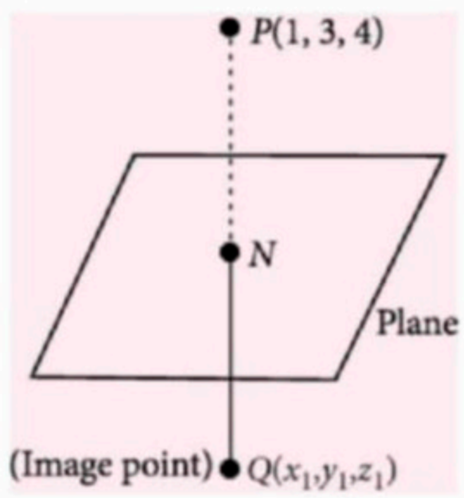
$$3(3r) + 4(4r) - 6(-6r) + 1 = 0 \Rightarrow r=-\frac{1}{61}$$

$\therefore$ Coordinates of the foot of the perpendicular $P$ are $\left(\frac{-3}{61}, \frac{-4}{61}, \frac{6}{61}\right)$.

**4. (a) :** As it is clear from the figure that $PQ$ will be perpendicular to the plane and foot of this perpendicular is mid-point of $PQ$ *i.e.*, $N$.

So, direction ratios of line $PQ$ are 2, –1, 1.

$\Rightarrow$ Equation of line




$PQ$ is $\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}=r$ (say)

Any point on line $PQ$ is $(2r + 1, -r + 3, r + 4)$

If this point lies on the plane, then

$2(2r + 1) - (-r + 3) + (r + 4) + 3 = 0 \Rightarrow r = -1$

$\therefore$ Coordinates of foot of perpendicular $N = (-1, 4, 3)$

As $N$ is middle point of $PQ$

$\therefore \quad -1=\frac{1+x_1}{2}, 4=\frac{3+y_1}{2}, 3=\frac{4+z_1}{2}$

$\Rightarrow x_1 = -3, y_1 = 5, z_1 = 2$

$\therefore$ Image of point $P(1, 3, 4)$ is the point $Q(-3, 5, 2)$.

**5. (b) :** Let the equation of sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d_1 = 0$

As sphere passes through points (1, 0, 0), (0, 1, 0) and (0, 0, 1). So, we have

$1 + 2u + d_1 = 0,\ 1 + 2v + d_1 = 0,\ 1 + 2w + d_1 = 0$

On solving, we get

$$u = v = w = -\frac{1}{2}(d_1 + 1)$$

If $r$ is the radius of the sphere, then

$$r = \sqrt{u^2 + v^2 + w^2 - d_1}$$

$$r^2 = \frac{3}{4}(d_1 + 1)^2 - d_1 = \mu \text{ (say)}$$

For $r$ to be minimum,

$$\frac{d\mu}{dd_1} = 0 \Rightarrow \frac{3}{4} \cdot 2(d_1 + 1) - 1 = 0 \Rightarrow d_1 = -\frac{1}{3}$$

Also, $\frac{d^2\mu}{dd_1^2} = \frac{3}{2} =$ positive at $d_1 = -\frac{1}{3}$

Hence, $r$ is minimum at $d_1 = -\frac{1}{3}$

So, substituting value of $d_1$, we have $u = v = w = -\frac{1}{3}$

$\therefore$ Equation of the sphere

$$x^2 + y^2 + z^2 - \frac{2}{3}(x + y + z) - \frac{1}{3} = 0$$

$\Rightarrow 3(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0$

**6. (d) :** Let the coordinates of moving point $P$ be $(x, y, z)$. Let $A(a, 0, 0)$ and $B(-a, 0, 0)$ be two fixed points. According to given condition

$$\frac{AP}{BP} = \text{constant} = k \text{ (say)} \Rightarrow AP^2 = k^2 BP^2$$

$\Rightarrow (x - a)^2 + (y - 0)^2 + (z - 0)^2$

$= k^2\{(x + a)^2 + (y - 0)^2 + (z - 0)^2\}$

$\Rightarrow (1 - k^2)(x^2 + y^2 + z^2) - 2ax(1 + k^2) + a^2(1 - k^2) = 0$

$\therefore$ Required locus is

$$x^2 + y^2 + z^2 - \frac{2a(1+k^2)}{(1-k^2)}x + a^2 = 0, \text{ which is a}$$

sphere.

**7. (a) :** Let the equation to the plane be $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$

$\Rightarrow \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 1$ ($\because$ Plane passes through $a, b, c$)

Now, the points of intersection of the plane with the coordinate axes are $A(\alpha, 0, 0)$, $B(0, \beta, 0)$ and $C(0, 0, \gamma)$.

$\Rightarrow$ Equation of planes parallel to the coordinate planes and passing through $A$, $B$ and $C$ are $x = \alpha$, $y = \beta$ and $z = \gamma$.

$\therefore$ The locus of the common point is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$

(By eliminating $\alpha, \beta, \gamma$ from above equation)

$\Rightarrow ayz + bzx + cxy = xyz.$

**8. (c) :** Let direction cosines of straight line be $l$, $m$ and $n$.

$\therefore 4l + m + n = 0;\quad l - 2m + n = 0$

$$\Rightarrow \frac{l}{3} = \frac{m}{-3} = \frac{n}{-9} \Rightarrow \frac{l}{-1} = \frac{m}{1} = \frac{n}{3}$$

$\therefore$ Equation of straight line is $\frac{x-2}{-1} = \frac{y+1}{1} = \frac{z+1}{3}$.

**9. (b) :** Let direction ratios of the required line be $< a, b, c >$

So, $a - 2b - 2c = 0$ and $2b + c = 0 \Rightarrow c = -2b$

Also, $a - 2b + 4b = 0 \Rightarrow a = -2b$

$\therefore$ Direction ratios of the required line are

$< -2b, b, -2b > = <2, -1, 2>$

Hence, direction cosines of the required line are

$$\left(\frac{2}{\sqrt{2^2 + 1^2 + 2^2}}, \frac{-1}{\sqrt{2^2 + 1^2 + 2^2}}, \frac{2}{\sqrt{2^2 + 1^2 + 2^2}}\right)$$

$$= \left(\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}\right)$$

**10. (a) :** Let $A \equiv (4, 7, 8)$, $B \equiv (2, 3, 4)$, $C \equiv (-1, -2, 1)$ and $D \equiv (1, 2, 5)$

Direction cosines of $AB$ are $\left(\frac{2}{6}, \frac{4}{6}, \frac{4}{6}\right) = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

Direction cosines of $CD$ are $\left(\frac{2}{6}, \frac{4}{6}, \frac{4}{6}\right) = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

So, $AB$ is parallel to $CD$.

Direction cosines of $AD$ are $\left(\frac{3}{\sqrt{43}}, \frac{5}{\sqrt{43}}, \frac{3}{\sqrt{43}}\right)$

Direction cosines of $BC$ are $\left(\frac{3}{\sqrt{43}}, \frac{5}{\sqrt{43}}, \frac{3}{\sqrt{43}}\right)$

So, $AD$ is parallel to $BC$.

Hence, $ABCD$ is a parallelogram.

**11. (b) :** Mid-point of $A(-2, 5, 1)$ and $B(6, 1, 5)$ is $(2, 3, 3)$

Direction ratios of the line joining $A$ and $B$ are $<2, -1, 1>$.

Therefore, equation of the plane perpendicular to $AB$ and passing through $(2, 3, 3)$ is

$2(x - 2) - 1(y - 3) + 1(z - 3) = 0 \Rightarrow 2x - y + z = 4$

**12. (b) :** $\theta = \cos^{-1}\left|\dfrac{\frac{1}{2\sqrt{3}} - \frac{1}{2\sqrt{3}} - \frac{1}{\sqrt{6}}}{\sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{2}}\sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}}}\right|$

$$= \cos^{-1}\left|\frac{-1}{\sqrt{6}}\right| = \cos^{-1}\frac{1}{\sqrt{6}}$$

**13. (a) :** The plane $ax + by + cz + d = 0$ intersects $x$ and $y$ axes at equal angles.
Therefore $|\cos\alpha| = |\cos\beta|$
$\Rightarrow \quad |a| = |b|$

**14. (c) :** We have,
$\Rightarrow \quad \cos^2 45° + m^2 + \cos^2 60° = 1$

$$\Rightarrow \quad m^2 = \frac{1}{4} \quad \Rightarrow \quad m = \pm\frac{1}{2}$$

$\therefore$ Angle made by line to the $y$-axis is $60°$.

**15. (b) :** Direction cosines of the line joining $(7, 9, 4)$ and $(4, 5, -8)$ are $< \frac{3}{13}, \frac{4}{13}, \frac{12}{13} >$
Therefore, required projection is

$$\frac{3}{13}(6) + \frac{4}{13}(5) + \frac{12}{13}(-1) = \frac{26}{13} = 2$$

**16. (b) :** Direction ratios of $AB$ are $< 2, -3, 5 >$.
Thus, equation of $AB$ is,

$$\frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-1}{5}$$

For the point of intersection of this line with $xy$-plane, we have $z = 0$

$$\Rightarrow \quad \frac{x-3}{2} = \frac{y-4}{-3} = \frac{-1}{5}$$

$$\Rightarrow \quad x = 3 - \frac{2}{5} = \frac{13}{5}, \; y = 4 + \frac{3}{5} = \frac{23}{5}$$

Hence, the required point is $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$

**17. (d) :** Let the direction cosine of the required line be $l$, $m$ and $n$.
We have, $ll_1 + mm_1 + nn_1 = 0$, $ll_2 + mm_2 + nn_2 = 0$

$$\Rightarrow \quad \frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

$$= \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{\Sigma(m_1n_2 - m_2n_1)^2}}$$

Also, $l_1l_2 + m_1m_2 + n_1n_2 = 0$
Thus, $\Sigma(m_1n_2 - m_2n_1)^2 = (\Sigma l_1)^2(\Sigma l_2)^2 - (\Sigma l_1l_2)^2 = 1$
$\Rightarrow \quad l = m_1n_2 - m_2n_1$, $m = n_1l_2 - n_2l_1$ and $n = l_1m_2 - l_2m_1$

**18. (c) :** If '$\theta$' be the angle between the plane and $x$–axis,

then $\sin\theta = \frac{2}{\sqrt{4+9+36}} = \frac{2}{7} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{2}{7}\right)$

$$\Rightarrow \quad a = \frac{2}{7}$$

**19. (b) :** Let the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
It passes through $(1, 0, 0)$ and $(0, 1, 0)$

$$\therefore \quad \frac{1}{a} = 1, \frac{1}{b} = 1 \quad \Rightarrow \quad a = b = 1$$

$$\text{Also, } \cos\frac{\pi}{4} = \frac{\left|\frac{1}{a} + \frac{1}{b}\right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}\sqrt{1+1}} \quad \Rightarrow \quad c = \pm\frac{1}{\sqrt{2}}$$

Thus, direction ratios are $(1, 1, \sqrt{2})$ or $(1, 1, -\sqrt{2})$

**20. (b) :** Plane will be of the form
$(x + y + z - 5) + a(2x - y + 3z - 1) = 0$
$\Rightarrow x(1 + 2a) + y(1 - a) + z(1 + 3a) = 5 + a$
It is parallel to the line $y = z = 0$.
$\therefore \quad (1 + 2a) = 0 \Rightarrow a = -1/2$
Thus, required equation of plane is

$$\frac{3}{2}y - \frac{1}{2}z = \frac{9}{2} \quad \Rightarrow \quad 3y - z = 9$$

**21. (d) :** Given line is $2x = 3y = 4z \Rightarrow \quad \frac{x}{6} = \frac{y}{4} = \frac{z}{3}$
Let the plane be $ax + by + cz = 1$
So, we have, $6a + 4b + 3c = 0$, $2a - b = 1$ and
$3a - 4b + 5c = 1$

$$\Rightarrow \quad a = \frac{29}{85}, b = -\frac{27}{85} \text{ and } c = -\frac{22}{85}$$

Thus, equation of plane is $29x - 27y - 22z = 85$

**22. (d) :** Any point on the given lines can be taken as
$P_1 = (r_1 + 2, r_1 + 3, -kr_1 + 4)$ and
$P_2 = (kr_2 + 1, 2r_2 + 4, r_2 + 5)$
These lines will intersect if for some $r_1$ and $r_2$, we have
$r_1 + 2 = kr_2 + 1$ ...(i), $r_1 + 3 = 2r_2 + 4$ ...(ii)
and $-kr_1 + 4 = r_2 + 5$ ...(iii)
From (i) and (ii), we have $r_1 - kr_2 + 1 = 0$, $r_1 = 2r_2 + 1$

$$\Rightarrow \quad r_2 = \frac{2}{k-2}, r_1 = \frac{k+2}{k-2}$$

Putting these values in (iii), we get
$k^2 + 3k = 0 \quad \Rightarrow \quad k \in \{0, -3\}$

**23. (a, c) :** For the point where the line intersects the curve, we have $z = 0$

$$\therefore \quad \frac{x-2}{3} = \frac{y-1}{2} = \frac{0-1}{-1} \Rightarrow x = 5 \text{ and } y = 3$$

Putting these values in $x^2 - y^2 = a^2$, we get $a^2 = 16$
$\Rightarrow \quad a = \pm 4$

**24. (a, b, c, d) :** Since, planes are parallel.

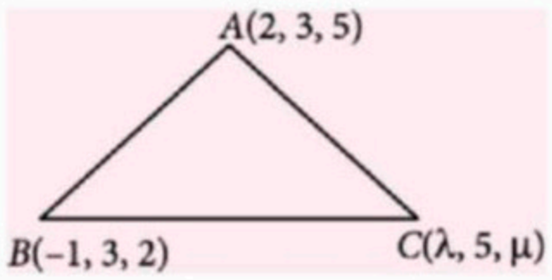
$$\therefore \quad P_1 = \frac{4}{\sqrt{29}}, P_2 = \frac{1}{2\sqrt{29}}, P_3 = \frac{8}{\sqrt{29}}$$

So, $P_3 = 16P_2 = 2P_1$, $P_1 + 2P_2 + 3P_3 = \sqrt{29}$ and
$P_1 + 8P_2 - P_3 = 0$

**25. (c, d) :** Required plane is given by $3x + 4y + \lambda(z) = 0$
Now, as the angle between $3x + 4y = 0$ and $3x + 4y + \lambda z = 0$ is $\pi/4$, therefore we get $\lambda = \pm 5$

**26. (a, b) :** Mid-point of $BC = \left(\frac{\lambda-1}{2}, 4, \frac{\mu+2}{2}\right)$

Direction ratios of median through $A$ are




$$\left(\frac{\lambda-1}{2}-2,\ 4-3,\ \frac{\mu+2}{2}-5\right)=\left(\frac{\lambda-5}{2},\ 1,\ \frac{\mu-8}{2}\right)$$

$\because$ The median through $A$ is equally inclined to axes, so the direction ratios must be equal

$$\therefore \quad \frac{\lambda-5}{2}=1=\frac{\mu-8}{2} \Rightarrow \lambda=7, \mu=10$$

**27. (a) :** $(3l + 4m - 5n)^2 \le (3^2 + 4^2 + (-5)^2)(l^2 + m^2 + n^2)$
$= 50 \times 1$

**28. (b) :** $\frac{l}{3}=\frac{m}{4}=\frac{n}{-5}=\frac{3l+4m-5n}{3^2+4^2+(-5)^2}=\frac{5\sqrt{2}}{50}=\frac{1}{5\sqrt{2}}$

$$\therefore \quad l=\frac{3}{5\sqrt{2}},\ m=\frac{4}{5\sqrt{2}} \text{ and } n=-\frac{1}{\sqrt{2}}$$

$$\therefore \quad |lm + mn + nl| = 0.46$$

**29. (a) :** $(ax + by + cz)^2$

$$=(\sqrt{a}\cdot\sqrt{a}x+\sqrt{b}\cdot\sqrt{b}\ y+\sqrt{c}\cdot\sqrt{c}\ z)^2$$

$$\le (a+b+c)(ax^2+by^2+cz^2)$$

$\Rightarrow \quad a + b + c \le (a + b + c)(ax^2 + by^2 + cz^2)$
$\Rightarrow \quad ax^2 + by^2 + cz^2 \ge 1$

**30. (b) :** Equation of the second plane is
$-x + 2y - 3z + 5 = 0$
$2(-1) + 3\cdot2 + (-4)(-3) > 0$
$\therefore$ Origin lies in obtuse angle.
Also, $(2 \times 1 + 3(-2) - 4 \times 3 + 7)(-1 + 2(-2) - 3 \times 3 + 5)$
$= (2 - 6 - 12 + 7)(-1 - 4 - 9 + 5) > 0$
$\therefore$ $P$ lies in obtuse angle.

**31. (c) :** $1 \times 2 + 2 \times 1 - 3 \times 3 < 0$
$\therefore$ Origin lies in acute angle.
Also, $(2 + 2(-1) - 3(2) + 5)(2 \times 2 - 1 + 3 \times 2 + 1)$
$= (-1)(10) < 0$
$\therefore$ $P$ lies in obtuse angle.

**32. (a) :** $1 - 4 - 9 < 0$
$\therefore$ Origin lies in acute angle.
Further, $(1 + 4 - 6 + 2)(1 - 4 + 6 + 7) = 10 > 0$
$\therefore$ $P$ lies in acute angle.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**33. A → s, t; B → p, t; C → q; D → r**

**(A)** Both the lines pass through the point (7, 11, 15)

**(B)** $< 2, 3, 4 >$ are direction ratios of both the lines.

Also, the point (1, 2, 3) is common to both.

$\therefore$ The lines are coincident.

**(C)** $< 5, 4 - 2 >$ are direction ratios of both the lines.

$\therefore$ The lines are parallel.

Also, $x = 2 + 5\lambda$, $y = -3 + 4\lambda$, $z = 5 - 2\lambda$

$$\therefore \quad \frac{2+5\lambda-7}{5}=\frac{-3+4\lambda-1}{4}=\frac{5-2\lambda-2}{-2}$$

$$\Rightarrow \quad \lambda-1=\lambda-1=\frac{3-2\lambda}{-2} \quad \therefore \text{ No value of } \lambda \text{ exist}$$

Thus, the lines are parallel and different.

**(D)** $< 2, 3, 5>$ and $< 3, 2, 5 >$ are direction ratios of first and second line respectively.

$\therefore$ The lines are not parallel.

$x = 3 + 2\lambda$, $y = -2 + 3\lambda$, $z = 4 + 5\lambda$

$x = 3 + 3\mu$, $y = 2 + 2\mu$, $z = 7 + 5\mu$

are parametric equations of the lines.

Solving $3 + 2\lambda = 3 + 3\mu$ and $-2 + 3\lambda = 2 + 2\mu$,

We get $\lambda=\frac{12}{5}, \mu=\frac{8}{5}$

Now, substituting these values in $4 + 5\lambda = 7 + 5\mu$

We get $4 + 12 = 7 + 8$

$\Rightarrow$ $16 = 15$, which is not true

$\therefore$ The lines do not intersect.

Hence, the lines are skew.

**34. A → s; B → p; C → r; D → q**

**(A)** Let the coordinates of foot of perpendicular $q$ be $(2 + 2\lambda, 1 + 3\lambda, 2 + 4\lambda)$

Direction ratios of $P(1, 2, 3)$ and $Q(2 + 2\lambda, 1 + 3\lambda, 2 + 4\lambda)$ are $< 2\lambda + 1, 3\lambda - 1, 4\lambda - 1 >$

$\therefore$ $2(2\lambda + 1) + 3(3\lambda - 1) + 4(4\lambda - 1) = 0$

$$\Rightarrow \quad 29\lambda=5 \Rightarrow \lambda=\frac{5}{29}$$

$\therefore$ Coordinates of foot of perpendicular $Q$ are $\left(\frac{68}{29}, \frac{44}{29}, \frac{78}{29}\right)$

**(B)** Let the image be the point $(a, b, c)$, then

$$\frac{1+a}{2}=\frac{68}{29}, \frac{2+b}{2}=\frac{44}{29} \text{ and } \frac{3+c}{2}=\frac{78}{29}$$

$$i.e., a=\frac{107}{29}, b=\frac{30}{29} \text{ and } c=\frac{69}{29}$$

**(C)** Equation of line passing through (2, 3, 5) and perpendicular to given plane is

$$\frac{x-2}{2}=\frac{y-3}{3}=\frac{z-5}{-4}=\frac{-10}{29} \therefore x=\frac{38}{29}, y=\frac{57}{29} \text{ and } z=\frac{185}{29}$$

Required coordinates are $\left(\frac{38}{29}, \frac{57}{29}, \frac{185}{29}\right)$

**(D)** $\because$ $$\frac{x-2}{3}=\frac{y-5}{-2}=\frac{z-1}{4}=-2\frac{(6-10+4-5)}{29}=\frac{10}{29}$$

$$x=2+\frac{30}{29}=\frac{88}{29}, y=5-\frac{20}{29}=\frac{125}{29}, z=1+\frac{40}{29}=\frac{69}{29}$$

$\therefore$ Image of point (2, 5, 1) in the given plane is given by $(x, y, z)$ *i.e.*, $\left(\frac{88}{29}, \frac{125}{29}, \frac{69}{29}\right)$

**35. A → r ; B → p ; C → q ; D → s**

Given $abc/6 = 32$, where $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively are the coordinates of $A$, $B$ and $C$.

**(A)** Centroid of tetrahedron

$$(\alpha,\beta,\gamma)=\left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right) \Rightarrow a=4\alpha, b=4\beta, c=4\gamma$$

$\therefore$ $64\alpha\beta\gamma = 32 \times 6 \Rightarrow \alpha\beta\gamma = 3$

**(B)** Equidistant point

$$(\alpha,\beta,\gamma)=\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right) \Rightarrow a=2\alpha, b=2\beta, c=2\gamma$$

$\therefore$ $8\alpha\beta\gamma = 32 \times 6 \Rightarrow \alpha\beta\gamma = 24$

**(C)** The equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$\therefore$ Foot of perpendicular from the origin is

$$(\alpha,\beta,\gamma)=\left(\frac{1/a}{\sum\frac{1}{a^2}}, \frac{1/b}{\sum\frac{1}{a^2}}, \frac{1/c}{\sum\frac{1}{a^2}}\right)$$

$$\Rightarrow \quad \frac{1}{a\alpha}=\frac{1}{b\beta}=\frac{1}{c\gamma}=t, \text{ where } t=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\sum\frac{1}{a^2}$$

$$\Rightarrow \quad t=(\alpha^2+\beta^2+\gamma^2)t^2 \Rightarrow t=\frac{1}{\alpha^2+\beta^2+\gamma^2} \text{ and}$$

$$a=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}, b=\frac{\alpha^2+\beta^2+\gamma^2}{\beta}, c=\frac{\alpha^2+\beta^2+\gamma^2}{\gamma}$$

Now $abc = 6 \times 32 \Rightarrow (\alpha^2 + \beta^2 + \gamma^2)^3 = 192\ \alpha\beta\gamma$

**(D)** Let $P$ be $(\alpha, \beta, \gamma)$, then $PA \perp PB$

$\Rightarrow$ $\alpha(\alpha - a) + \beta(\beta - b) + \gamma\gamma = 0$

$\Rightarrow$ $a\alpha + b\beta = \alpha^2 + \beta^2 + \gamma^2$

Similarly, $PB \perp PC \Rightarrow \alpha\alpha + \beta(\beta - b) + \gamma(\gamma - c) = 0$

$\Rightarrow$ $b\beta + c\gamma = \alpha^2 + \beta^2 + \gamma^2$

Also, $PA \perp PC \Rightarrow \alpha(\alpha - a) + \beta^2 + \gamma(\gamma - c) = 0$

$\Rightarrow$ $a\,\alpha + c\,\gamma = \alpha^2 + \beta^2 + \gamma^2$

$$\therefore a=\frac{\alpha^2+\beta^2+\gamma^2}{2\alpha}, \ b=\frac{\alpha^2+\beta^2+\gamma^2}{2\beta}, c=\frac{\alpha^2+\beta^2+\gamma^2}{2\gamma}$$

Now, $abc = 6 \times 32$

$\Rightarrow$ $(\alpha^2 + \beta^2 + \gamma^2)^3 = 192 \times 8\ \alpha\beta\gamma = 1536\ \alpha\beta\gamma$

# 7 TIPS
## TO EXCEL YOURSELF AT MATHS

Getting good at Maths is not always a question of natural ability. There are various approaches you can take to your maths studies, all of which can have some real positive effects and outcomes.

**Here are 7 tips to overcome maths phobia and excel yourself.**

**1 Choose the right methodology**

Much like following a routine, how you prepare and how you tackle maths are as important as how good you are at it. A very young student counting whole numbers and adding and subtracting obviously won't take the same approach as older students looking at exponent, differential, quadratic or linear equations and complex numbers, so be sure to tailor your revision methods suitably to your level and course. There are many references that you can follow on a number of online websites many of which offer certified courses that will help you to get better at maths with just a little self-study. So, rather than worrying about where to start, switch on your computer or laptop and give it a whirl.

**2 Understand first and learn maths second**

This can sound like a bit of a paradox when we're talking about maths. However, it's useless learning everything off by heart if you don't have a clue how to use what you've learned. It's great to learn dozens of formulae but if you don't know what they are used for and how to apply them, it'll all be for nothing. If you can't manage simple algebraic formulae, don't try solving a polynomial equation. There's a reason why your modules are organised in a specific way.

**3 Don't look at the answer first**

It's tempting but you must not ever look at the answers. (Just as we've mentioned above, don't allow yourself to check your peer's work before reaching the answer yourself, you'll only be working backward from there which will end in disaster.)

Getting stuck on an exercise doesn't mean you are stupid and certainly isn't anything to be ashamed of. Spending time tackling a problem can be one of the best ways to learn maths and truly understand it. In maths, it's not all about getting to the answer immediately.

**4 Learn new habits to improve your maths skills**

To get good at maths, it's important to not get distracted. Always focus on what you have to do and solely on the page in front of you. It is no good having one eye on a game or magazine while doing your maths homework or revision, your brain needs to think exclusively about math. Before you set out on doing some maths related work, get rid of any major distractions, such as your mobile phone.

No texting your friend saying "I'm totally stuck on Question 3 of this maths prep" it is NOT classed as thinking about maths. This subject is one that you really need to give your undivided attention. So, take your time and make a habit of fully completing every tutorial in a calming environment; paying the attention means you're already halfway there.

**5 Practice makes perfect when learning maths**

Like we said before, if you want to become good at maths, you must keep working until you've completely understood your lesson. If you're the student, going to classes isn't always enough. Whether you're at school or taking night classes, you must regularly practice maths at home. It might even be more worthwhile if you can check out the online websites offering this in India.

**6 Always ask questions**

In maths, more than any other subject, curiosity pays off. As soon as you have any doubts, are confused, or something isn't clear, make sure you ask why. The phrasing of one person may be much clearer than that of another. You might understand it better just because they used different expressions.

**7 Learn to avoid the common maths mistakes**

To get good at maths, you don't always have to take classes, you can also learn to avoid common errors. Focusing on how to avoid these common mistakes is a great way to improve.

**(a) Overconfidence in your own abilities :** Overconfidence can bias our own judgements, individuals should have a balanced perception of their abilities.

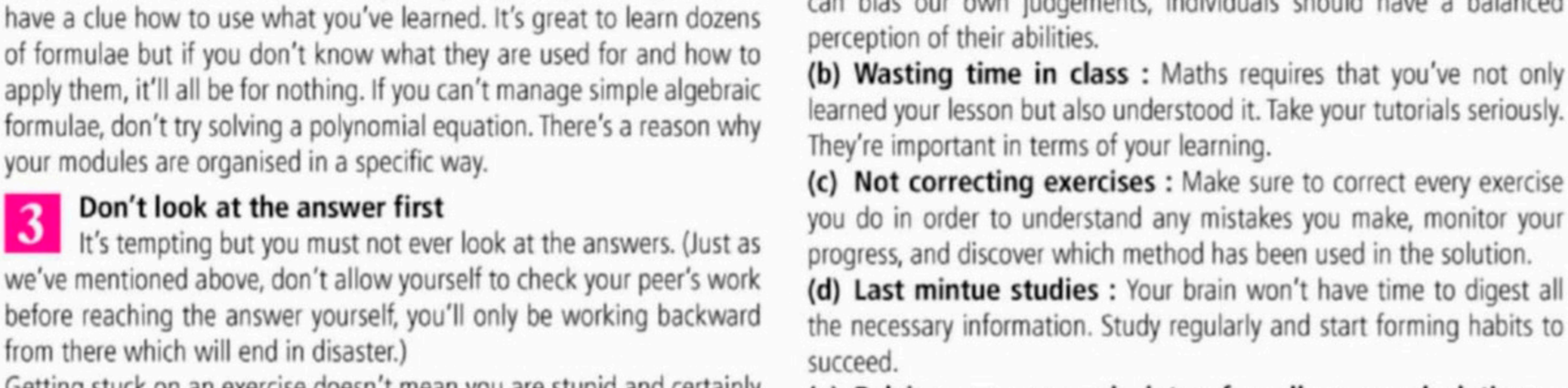
**(b) Wasting time in class :** Maths requires that you've not only learned your lesson but also understood it. Take your tutorials seriously. They're important in terms of your learning.

**(c) Not correcting exercises :** Make sure to correct every exercise you do in order to understand any mistakes you make, monitor your progress, and discover which method has been used in the solution.

**(d) Last mintue studies :** Your brain won't have time to digest all the necessary information. Study regularly and start forming habits to succeed.

**(e) Relying on your calculator for all your calculations :** A calculator is just a tool; you need to use it alongside your own knowledge. The result it gives you is dependent on how you put in the information. You should focus on mastering the basic concepts of mental arithmetic. You should also look into learning how to count quickly.

**All the Best!**

# MONTHLY TEST DRIVE

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

**Total Marks : 80** **Time Taken : 60 Min.**

## Series 9 : Vector Algebra and Three Dimensional Geometry

### Only One Option Correct Type

**1.** If $(\vec{a}\times\vec{b})^2+(\vec{a}\cdot\vec{b})^2=144$ and $|\vec{a}|=4$, then $|\vec{b}|$ is equal to
(a) 16 (b) 8 (c) 3 (d) 12

**2.** The value of $x$ for which the angle between $\vec{a}=2x^2\hat{i}+4x\hat{j}+\hat{k}$ and $\vec{b}=7\hat{i}-2\hat{j}+x\hat{k}$ is obtuse and the angle between $\vec{b}$ and the $z$-axis is acute and less than $\pi/6$, are
(a) $0<x<1/2$ (b) $1/2<x<15$
(c) $x>1/2$ or $x<0$ (d) none of these

**3.** The distance of the point (1, 0, 2) from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x-y+z=16$, is
(a) $3\sqrt{21}$ (b) 13 (c) $2\sqrt{14}$ (d) 8

**4.** Let $\vec{p}$ and $\vec{q}$ be the position vectors of $P$ and $Q$ respectively, with respect to $O$ and $|\vec{p}|=p$, $|\vec{q}|=q$. The points $R$ and $S$ divide $PQ$ internally and externally in the ratio 2 : 3 respectively. If $OR$ and $OS$ are perpendiculars then
(a) $9p^2=4q^2$ (b) $4p^2=9q^2$
(c) $9p=4q$ (d) $4p=9q$

**5.** The distance of the point (1, 3, –7) from the plane passing through the point (1, –1, –1), having normal perpendicular to both the lines $\frac{x-1}{1}=\frac{y+2}{-2}=\frac{z-4}{3}$ and $\frac{x-2}{2}=\frac{y+1}{-1}=\frac{z+7}{-1}$, is
(a) $\frac{20}{\sqrt{74}}$ (b) $\frac{10}{\sqrt{83}}$
(c) $\frac{10}{\sqrt{74}}$ (d) $\frac{5}{\sqrt{83}}$

**6.** The image of the line $\frac{x-1}{3}=\frac{y-3}{1}=\frac{z-4}{-5}$ in the plane $2x-y+z+3=0$ is the line
(a) $\frac{x-3}{3}=\frac{y+5}{1}=\frac{z-2}{-5}$ (b) $\frac{x-3}{-3}=\frac{y+5}{-1}=\frac{z-2}{5}$
(c) $\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$ (d) $\frac{x+3}{-3}=\frac{y-5}{-1}=\frac{z+2}{5}$

### One or More Than One Option(s) Correct Type

**7.** If the resultant of three forces $\vec{F}_1=p\hat{i}+3\hat{j}-\hat{k}$, $\vec{F}_2=6\hat{i}-\hat{k}$ and $\vec{F}_3=-5\hat{i}+\hat{j}+2\hat{k}$ acting on a particle has a magnitude equal to 5 units, then the value of $p$ is
(a) –6 (b) –4 (c) 2 (d) 4

**8.** If $\vec{a}$ and $\vec{b}$ are unequal unit vectors such that $(\vec{a}-\vec{b})\times[(\vec{b}+\vec{a})\times(2\vec{a}+\vec{b})]=\vec{a}+\vec{b}$, then angle $\theta$ between $\vec{a}$ and $\vec{b}$ is
(a) 0 (b) $\pi/2$ (c) $\pi/4$ (d) $\pi$

**9.** If the volume of tetrahedron $ABCD$ is 1 cubic units, where $A(0, 1, 2)$, $B(-1, 2, 1)$ and $C(1, 2, 1)$, then the locus of point $D$ is
(a) $x+y-z=3$ (b) $y+z=6$
(c) $y+z=0$ (d) $y+z=-3$

**10.** A parallelogram is constructed on vectors $\vec{a}=3\vec{\alpha}-\vec{\beta}$, $\vec{b}=\vec{\alpha}+3\vec{\beta}$, if $|\vec{\alpha}|=|\vec{\beta}|=2$ and angle between $\vec{\alpha}$ and $\vec{\beta}$ is $\frac{\pi}{3}$, then the length of the diagonal of parallelogram is
(a) $4\sqrt{5}$ (b) $4\sqrt{3}$
(c) $4\sqrt{7}$ (d) none of these

MTG

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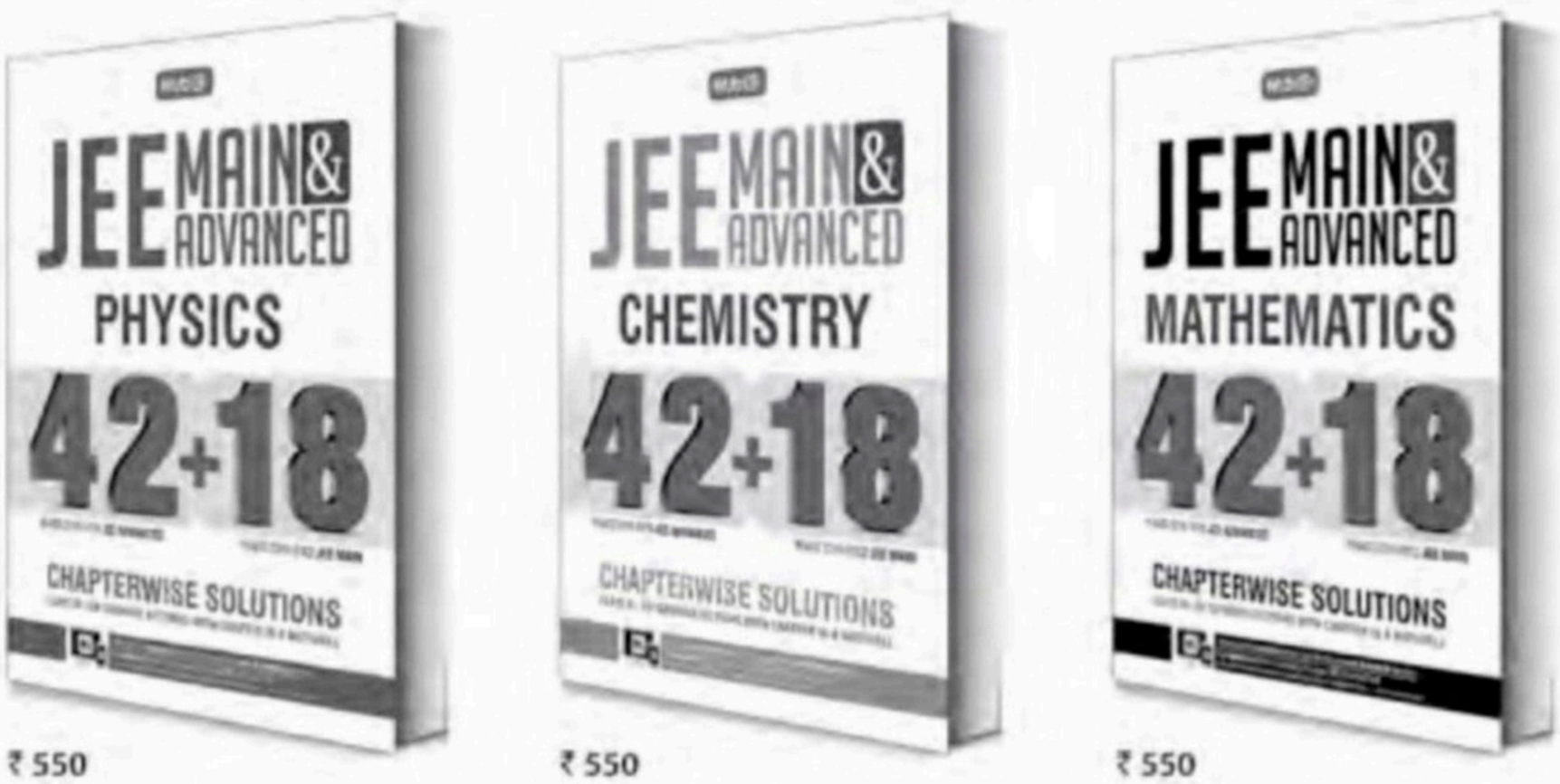


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**11.** Let $\vec{u}, \vec{v}, \vec{w}$ be three unit vectors such that $\vec{u}+\vec{v}+\vec{w}=\vec{a}$, $\vec{a}\cdot\vec{u}=\frac{3}{2}$, $\vec{a}\cdot\vec{v}=\frac{7}{4}$ and $|\vec{a}|=2$, then

(a) $\vec{u}\cdot\vec{v}=\frac{3}{4}$ (b) $\vec{v}\cdot\vec{w}=0$

(c) $\vec{u}\cdot\vec{w}=-\frac{1}{4}$ (d) none of these

**12.** The equations of the plane which passes through $(0, 0, 0)$ and which is equally inclined to the planes $x-y+z-3=0$ and $x+y+z+4=0$ is/are

(a) $y=0$ (b) $x=0$

(c) $x+y=0$ (d) $x+z=0$

**13.** Find the ratio in which the join of $(1, -2, 3)$ and $(4, 2, -1)$ is divided by the $XOY$ plane.

(a) 2 : 3 (b) 3 : 1

(c) 1 : 2 (d) 4 : 1

## Comprehension Type

Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $|\vec{a}|=|\vec{b}|=|\vec{c}|=4$ and angle between $\vec{a}$ and $\vec{b}$ is $\pi/3$, angle between $\vec{b}$ and $\vec{c}$ is $\pi/3$ and angle between $\vec{c}$ and $\vec{a}$ is $\pi/3$.

On the basis of above information, answer the following questions:

**14.** The volume of the parallelepiped whose adjacent edges are represented by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

(a) $24\sqrt{2}$ (b) $24\sqrt{3}$ (c) $32\sqrt{2}$ (d) $32\sqrt{3}$

**15.** The volume of the tetrahedron whose adjacent edges are represented by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

(a) $\frac{4\sqrt{3}}{2}$ (b) $\frac{8\sqrt{2}}{3}$ (c) $\frac{16}{\sqrt{3}}$ (d) $\frac{16\sqrt{2}}{3}$

## Matrix Match Type

**16.** Match the following columns :

| Column-I | | Column-II | |
|---|---|---|---|
| P. | If a line makes an angle $\alpha$ with $x$-axis and $y$-axis, then $\cot\alpha$ can be equal to | 1. | $-3$ |
| Q. | If the straight lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{4-z}{\lambda}$ and $\frac{x-1}{\lambda}=\frac{y-4}{2}=\frac{z-5}{1}$ intersect, then the value of $\lambda$ is | 2. | 2 |
| R. | If the plane $\lambda x-\mu y+\nu z=0$ contains line $\frac{x-\lambda}{\lambda}=\frac{y-2k}{\mu}=\frac{z-\nu}{\nu}$, then the value of $\frac{\mu}{k}, \mu\neq 0$ is | 3. | 0 |
| | | 4. | 1 |

| | P | Q | R |
|---|---|---|---|
| (a) | 2 | 3 | 1,4 |
| (b) | 3,4 | 1,4 | 3 |
| (c) | 3,4 | 1,3 | 2,3 |
| (d) | 3,4 | 1,3 | 2 |

## Numerical Value Type

**17.** Given a cube $ABCDA_1B_1C_1D_1$ with lower base $ABCD$, upper base $A_1B_1C_1D_1$ and the lateral edges $AA_1$, $BB_1$, $CC_1$ and $DD_1$; $M$ and $M_1$ are the centres of the faces $ABCD$ and $A_1B_1C_1D_1$ respectively. $O$ is a point on the line $MM_1$ such that $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=\overrightarrow{OM_1}$, then $\overrightarrow{OM}=\lambda\,\overrightarrow{OM_1}$ if $\lambda$ is equal to

**18.** The radius of the circle in which the sphere $x^2+y^2+z^2+2x-2y-4z-19=0$ is cut by the plane $x+2y+2z+7=0$ is

**19.** Let the position vectors of the points $P$ and $Q$ be $4\hat{i}+\hat{j}+\lambda\hat{k}$ and $2\hat{i}-\hat{j}+\lambda\hat{k}$, respectively. Vector $\hat{i}-\hat{j}+6\hat{k}$ is perpendicular to the plane containing the origin and the points $P$ and $Q$. Then $\lambda$ equals to

**20.** Let the equation of the plane containing line $x-y-z-4=0=x+y+2z-4$ and parallel to the line of intersection of the planes $2x+3y+z=1$ and $x+3y+2z=2$ be $x+Ay+Bz+C=0$. Then the value of $|A+B+C-4|$ is

*Keys are published in this issue. Search now!* ☺

## SELF CHECK

No. of questions attempted .....
No. of questions correct .....
Marks scored in percentage .....

Check your score! If your score is

| | | |
|---|---|---|
| > 90% | EXCELLENT WORK ! | You are well prepared to take the challenge of final exam. |
| 90-75% | GOOD WORK ! | You can score good in the final exam. |
| 74-60% | SATISFACTORY ! | You need to score more next time. |
| < 60% | NOT SATISFACTORY! | Revise thoroughly and strengthen your concepts. |

# CBSE warm-up!

Exam on 17th March 2020

CLASS-XII

**Practice questions for CBSE Exams as per the latest pattern and marking scheme issued by CBSE for the academic session 2019-20.**

## Practice Paper 2020

Time Allowed : 3 hours
Maximum Marks : 80

**GENERAL INSTRUCTIONS**

(i) All the questions are compulsory.
(ii) This question paper contains 36 questions.
(iii) Section A comprises of 20 questions of 1 mark each. Section B comprises of 6 questions of 2 marks each. Section C comprises of 6 questions of 4 marks each. Section D comprises of 4 questions of 6 marks each.
(iv) There is no overall choice. However, an internal choice has been provided in three questions of 1 mark each, two questions of 2 marks each, two questions of 4 marks each, and two questions of 6 marks each. You have to attempt only one of the alternatives in all such questions.
(v) Use of calculators is not permitted.

### SECTION - A

**Q.1-Q.10 are multiple choice type questions. Select the correct option.**

**1.** Find $x$ and $y$ respectively such that $\begin{bmatrix} x-y & 3 \\ 2x-y & 2x+1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 12 & 15 \end{bmatrix}$.

(a) 7, 2 (b) 2, 7 (c) 3, 4 (d) 2, 9

**2.** If $c_{ij}$ is the cofactor of the element $a_{ij}$ of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, then write the value of $a_{32} \cdot c_{32}$.

(a) 110 (b) 22 (c) −110 (d) −22

**3.** If $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=2\hat{i}-4\hat{k}, \vec{c}=\hat{i}+\lambda\hat{j}+3\hat{k}$ are coplanar, then the value of $\lambda$ is

(a) 5/2 (b) 3/5 (c) 5/3 (d) 2/5

**4.** The principal value of $\cot^{-1}(-\sqrt{3})$ is

(a) $5\pi/6$ (b) $\pi/3$ (c) $\pi/2$ (d) $\pi/4$

**5.** If the angle between the planes $2x - y + 2z = 3$ and $3x + 6y + cz = 4$ is $\cos^{-1}\left(\frac{4}{21}\right)$, then $c^2 =$

(a) 1 (b) 4 (c) 9 (d) 5

**6.** Evaluate : $\int 2^{2^{2^x}} 2^{2^x} 2^x \, dx$

(a) $\frac{1}{(\log 2)^3} 2^{2^{2^x}} + C$ (b) $\frac{1}{(\log 2)^3} 2^{2^x} + C$

(c) $\frac{1}{(\log 2)^2} 2^{2^x} + C$ (d) $\frac{1}{(\log 2)^4} 2^{2^{2^x}} + C$

**7.** Evaluate : $\int \frac{\cot x}{\sqrt[3]{\sin x}} dx$

(a) $\frac{-3}{\sqrt[3]{\sin x}} + C$ (b) $\frac{-2}{\sin^3 x} + C$

(c) $\frac{3}{\sin^{1/3} x}+C$ (d) None of these

**8.** Which of the following statements is false?

(a) The maximum (or minimum) solution of the objective function occurs at the vertex of the feasible region.

(b) The feasible region is always a concave region.

(c) If two corner points produce the same maximum (or minimum) value of the objective function, then every point on the line segment joining these points will also give the same maximum (or minimum) value.

(d) None of these

**9.** $\int xe^{x^2} dx$ is equal to

(a) $-\frac{e^{x^2}}{2}+C$ (b) $\frac{e^{x^2}}{2}+C$

(c) $\frac{e^x}{2}+C$ (d) $-\frac{e^x}{2}+C$

**10.** A bag contains 5 red and 3 blue balls. If 3 balls are drawn at random without replacement the probability of getting exactly one red ball is

(a) $\frac{45}{196}$ (b) $\frac{135}{392}$ (c) $\frac{15}{56}$ (d) $\frac{15}{29}$

**(Q.11-Q.15) Fill in the blanks.**

**11.** Let $R$ be the equivalence relation in the set $A = \{0, 1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : 2 \text{ divides } (a - b)\}$. The equivalence class of 0 = _____.

**12.** The amount of pollution content added in air in a city due to $x$ diesel vehicles is given by $P(x) = 0.005x^3 + 0.02x^2 + 30x$. The marginal increase in pollution content when 3 diesel vehicles are added, is _____.

**OR**

The function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always _____ on $R$.

**13.** For any square matrix $A$, $AA^T$ is a _____ matrix.

**14.** If $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}+2\hat{k}$, then the value of $\vec{a}\cdot(\vec{b}\times\vec{c})$ is _____.

**OR**

The magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is 9/2, is _____.

**15.** The value of $\lambda$, such that the line $\frac{x-2}{6}=\frac{y-1}{-2}=\frac{z+5}{\lambda}$ is perpendicular to the plane $3x - y - 2z = 7$ is _____.

**(Q.16-Q.20) Answer the following questions.**

**16.** Evaluate : $\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$.

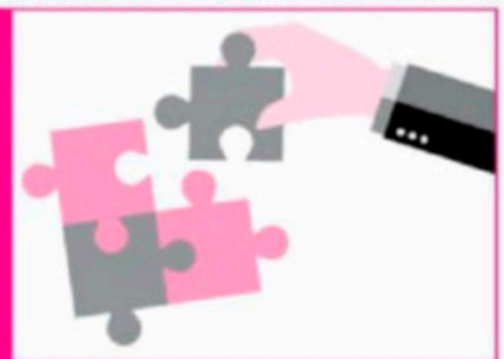
**17.** Suppose that five good fuses and two defective ones have been mixed up. To find the defective fuses, we test them one-by-one, at random and without replacement. What is the probability that we are lucky and find both of the defective fuses in the first two tests?

**18.** Evaluate : $\int_0^{\pi/2} x\cos x\, dx$

**OR**

Evaluate : $\int \cos^3 x \sin x\, dx$

# PUZZLE CORNER



## MATHDOKU

Introducing MATHDOKU, a mixture of ken-ken, sudoku and Mathematics. In this puzzle 6 × 6 grid is given, your objective is to fill the digits 1-6 so that each appear exactly once in each row and each column.

Notice that most boxes are part of a cluster. In the upper-left corner of each multibox cluster is a value that is combined using a specified operation on its numbers. For example, if that value is 3 for a two-box cluster and operation is multiply, you know that only 1 and 3 can go in there. But it is your job to determine which number goes where! A few cluster may have just one box and that is the number that fills that box.

| | | | | | |
|---|---|---|---|---|---|
| 4– | 24× | | | 8+ | |
| | 12× | | 20× | | |
| 1– | | | 11+ | | 3+ |
| 1– | | 120× | 2+ | | |
| 3+ | | | 36× | | |
| | | 11+ | | 2– | |

*Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.*

19. Differentiate the function $\cos(\sin x)$ w.r.t. $x$.

20. Determine the order and degree of

$$\frac{5d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}.$$

## SECTION - B

21. Simplify : $\tan^{-1}\left[\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right]$, if $\frac{a}{b}\tan x > -1$

**OR**

Let the function $f: R \to R$ be defined by $f(x) = x^2 + 5x + 9$, find $f^{-1}(8)$ and $f^{-1}(9)$.

22. Find the equation of normal to the curve $x^{2/3} + y^{2/3} = 2$ at $(1, 1)$.

23. If $A$ and $B$ are two independent events, then prove that the probability of occurrence of at least one of $A$ and $B$ is given by $1 - P(A')\,P(B')$.

24. Find the value of $p$, for which $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel vectors.

**OR**

Find the volume of the parallelepiped whose coterminous edges are represented by the vectors $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$.

25. Find the points on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ at a distance of 5 units from the point $P(1, 3, 3)$.

26. Determine the constants $a$ and $b$ such that the function $f(x) = \begin{cases} ax^2 + b, & \text{if } x > 2 \\ 2, & \text{if } x = 2 \\ 2ax - b, & \text{if } x < 2 \end{cases}$ is continuous at $x = 2$.

## SECTION - C

27. Let $f: N \to R$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f: N \to S$ is invertible, where $S$ is the range of $f$. Also, find inverse of $f$.

28. Find all points of discontinuity of $f$, where $f$ is defined as follows :

$$f(x) = \begin{cases} |x| + 3, & x \le -3 \\ -2x, & -3 < x < 3 \\ 6x + 2, & x \ge 3 \end{cases}$$

**OR**

For what value of $\lambda$, the function defined by $f(x) = \begin{cases} \lambda(x^2 + 2), & \text{if } x \le 0 \\ 4x + 6, & \text{if } x > 0 \end{cases}$ is continuous at $x = 0$?

Hence check the differentiability of $f(x)$ at $x = 0$.

29. Evaluate : $\int \frac{x^2 + x + 1}{(x^2 + 1)(x + 2)}\,dx$

30. Solve the following differential equation $y^2dx + (x^2 - xy + y^2)dy = 0$.

31. Two cards are drawn simultaneously (without replacement) from a well-shuffled pack of 52 cards. Find the probability distribution of the number of aces. Also, find the mean of the distribution.

**OR**

There are two bags, bag I and bag II. Bag I contains 4 white and 3 red balls while bag II contains 3 white and 7 red balls. One ball is drawn at random from one of the bags and it is found to be white. Find the probability that it was drawn from bag I.

32. A dietician wishes to mix two types of food in such a way that the vitamin content of the mixture contain at least 8 units of vitamin $A$ and 10 units of vitamin $C$. Food I contains 2 units per kg of vitamin $A$ and 1 unit per kg of vitamin $C$, while food II contain 1 unit per kg of vitamin $A$ and 2 units per kg of vitamin $C$. It cost ₹ 5 per kg to purchase food I and ₹ 7 per kg to purchase food II. Determine the minimum cost of the mixture. Formulate the LPP and solve it.

## SECTION - D

33. If $a$, $b$, $c$ are positive and unequal, show that the following determinant is negative.

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

**OR**

If $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix}$, find $A^{-1}$ and hence solve the system of equations :

$x + 2y + z = 4$, $-x + y + z = 0$, $x - 3y + z = 4$



| | | | |
|---|---|---|---|
| 1. (b) | 2. (d) | 3. (a) | 4. (a) |
| 5. (b) | 6. (b) | 7. (a, d) | 8. (a, b) |
| 9. (a, c) | 10. (a, c) | 11. (a, c) | 12. (a, b) |
| 13. (c, d) | 14. (b) | 15. (d) | 16. (d) |
| 17. (2.7) | 18. (–7) | 19. (1) | 20. (512) |

34. Using integration, find the area of the region enclosed between the two circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$.

35. Find the intervals in which the function $f$ given by $f(x) = \sin x - \cos x, 0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

**OR**

If the length of three sides of a trapezium other than base is 10 cm each, then find the area of the trapezium when it is maximum.

36. Find the equation of the plane that contains the point (1, –1, 2) and is perpendicular to both the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$. Hence find the distance of point $P(-2, 5, 5)$ from the plane obtained above.

## SOLUTIONS

**1. (a) :** We have, $\begin{bmatrix} x-y & 3 \\ 2x-y & 2x+1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 12 & 15 \end{bmatrix}$

$\Rightarrow x - y = 5$ ...(i)

$2x - y = 12$ ...(ii)

Subtracting (i) from (ii), we get $x = 7$

From (i), $y = x - 5 = 7 - 5 = 2$

**2. (a) :** Here, $a_{32} = 5$, then,

$$c_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = (-1)^5(8-30) = -(-22) = 22$$

$\therefore \quad a_{32} \cdot c_{32} = 5 \times 22 = 110$

**3. (c) :** Since, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - 4\hat{k}$, $\vec{c} = \hat{i} + \lambda\hat{j} + 3\hat{k}$ are coplanar.

$$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & -4 \\ 1 & \lambda & 3 \end{vmatrix} = 0$$

$$\Rightarrow 4\lambda - 1(6+4) + 2\lambda = 0 \Rightarrow \lambda = \frac{5}{3}$$

**4. (a) :** Let $\cot^{-1}(-\sqrt{3}) = \theta \Rightarrow \cot\theta = -\sqrt{3} = -\cot\frac{\pi}{6}$

$$= \cot\left(\pi - \frac{\pi}{6}\right) = \cot\frac{5\pi}{6} \Rightarrow \theta = \frac{5\pi}{6} \in (0, \pi)$$

$\therefore$ Principal value of $\cot^{-1}(-\sqrt{3})$ is $\frac{5\pi}{6}$.

**5. (b) :** D.r.'s of the normals are (2, –1, 2) and (3, 6, $c$).

According to question, $\theta = \cos^{-1}\left(\frac{4}{21}\right) \Rightarrow \cos\theta = \frac{4}{21}$

$$\Rightarrow \frac{6-6+2c}{3\sqrt{45+c^2}} = \frac{4}{21} \Rightarrow \frac{c^2}{(45+c^2)} = \frac{4}{49}$$

$\Rightarrow 45c^2 = 4 \times 45 \Rightarrow c^2 = 4$

**6. (a) :** Let $I = \int 2^{2^{2^x}} 2^{2^x} 2^x dx$

Put $2^{2^{2^x}} = t \Rightarrow 2^{2^{2^x}} 2^{2^x} 2^x (\log 2)^3 dx = dt$

$$\therefore \quad I = \int \frac{1}{(\log 2)^3} dt = \frac{1}{(\log 2)^3} t + C = \frac{1}{(\log 2)^3} 2^{2^{2^x}} + C$$

**7. (a) :** Let $I = \int \frac{\cot x}{\sqrt[3]{\sin x}} dx = \int \frac{\cos x}{\sin^{1/3} x \cdot \sin x} dx$

$$= \int \frac{\cos x}{\sin^{4/3} x} dx = \int \sin^{-4/3} x \cdot \cos x\, dx$$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\therefore \quad I = \int t^{-4/3}\, dt = \frac{t^{-1/3}}{-1/3} + C = \frac{-3}{\sqrt[3]{\sin x}} + C$$

**8. (b) :** The feasible region is always a convex region.

**9. (b) :** Let $I = \int xe^{x^2} dx$

Put $x^2 = t \Rightarrow 2xdx = dt \Rightarrow x\,dx = \frac{dt}{2}$

$$\therefore \quad I = \frac{1}{2}\int e^t dt = \frac{e^t}{2} + C = \frac{e^{x^2}}{2} + C$$

## SOLUTION SENDERS

**Mathdoku (December)**

- Varun Chaturvedi (Gujarat)

**Mathdoku (January)**

- Pravin Bajirao Thite (Maharashtra)
- Mukul Prajapat (Rajasthan)
- B. Aakash (Tamil Nadu)
- Sidak Singh Kalsi

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**Samurai Sudoku (December)**

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**Samurai Sudoku (January)**

- Pavithra T. Soman (Kerala)
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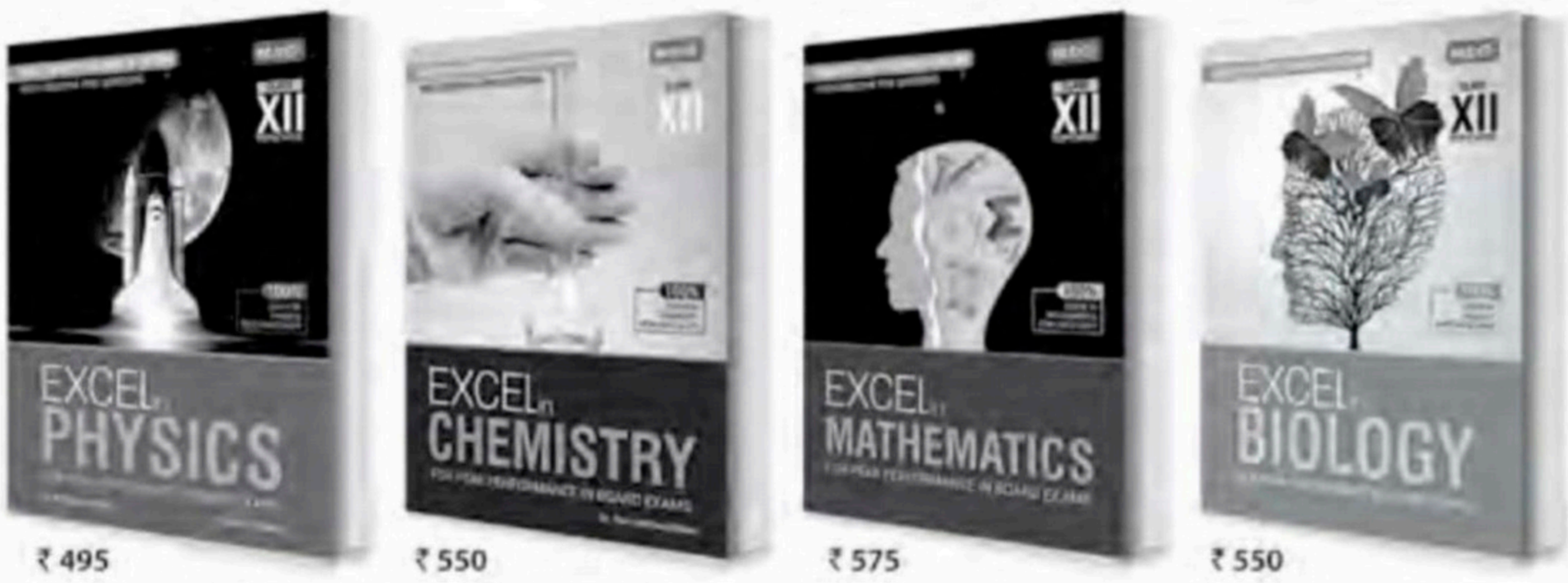
- Ayantika Pal (West Bengal)
- Kalur Nikhil (Telangana)
- Subhranil Pan (West Bengal)
- Pravin Bajirao Thite (Maharashtra)

# Concerned about your performance in Class XII Boards?

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**10. (c) :** Required probability = $P\{(RBB), (BRB), (BBR)\}$
$= P(RBB) + P(BRB) + P(BBR)$
$= \frac{5}{8}\times\frac{3}{7}\times\frac{2}{6}+\frac{3}{8}\times\frac{5}{7}\times\frac{2}{6}+\frac{3}{8}\times\frac{2}{7}\times\frac{5}{6} = 3\times\frac{5}{56} = \frac{15}{56}$

**11.** Here, $R = \{(a, b) \in A \times A : 2 \text{ divides } (a - b)\}$
This is the given equivalence relation, where
$A = \{0, 1, 2, 3, 4, 5\}$
$\therefore \quad [0] = \{0, 2, 4\}$

**12.** We have, $P(x) = 0.005x^3 + 0.02x^2 + 30x$
$\Rightarrow \frac{dP}{dx} = 0.015x^2 + 0.04x + 30$
Now, $\left(\frac{dP}{dx}\right)_{x=3} = 0.015\times3^2 + 0.04\times3 + 30$
$= 0.135 + 0.12 + 30 = 30.255$
Hence, marginal increase in pollution when $x = 3$ is 30.255.

**OR**

We have, $f(x) = 4x^3 - 18x^2 + 27x - 7$
$\Rightarrow f'(x) = 12x^2 - 36x + 27$
$= 12\left(x^2 - 3x + \frac{9}{4}\right) = 12\left(x - \frac{3}{2}\right)^2 \geq 0 \ \forall\, x \in R$
Hence, $f(x)$ is always increasing on $R$.

**13.** We have, $(AA^T)^T = (A^T)^T A^T = AA^T$
$\therefore \quad AA^T$ is a symmetric matrix.

**14.** Here, $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$,
$\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$

Now, $\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 3\hat{i} + 5\hat{j} - 7\hat{k}$

$\therefore \ \vec{a}\cdot(\vec{b}\times\vec{c}) = (2\hat{i} + \hat{j} + 3\hat{k})\cdot(3\hat{i} + 5\hat{j} - 7\hat{k})$
$= 2\times3 + 1\times5 + 3\times(-7) = 6 + 5 - 21 = -10$

**OR**

Given, $|\vec{a}| = |\vec{b}|, \theta = 60°$ and $\vec{a}\cdot\vec{b} = \frac{9}{2}$

Now, $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$\Rightarrow \cos 60° = \frac{9/2}{|\vec{a}|^2} \Rightarrow \frac{1}{2} = \frac{9/2}{|\vec{a}|^2}$

$\Rightarrow |\vec{a}|^2 = 9 \Rightarrow |\vec{a}| = 3 \ \therefore \ |\vec{a}| = |\vec{b}| = 3$

**15.** Direction ratios of line and normal to plane are $(6, -2, \lambda)$ and $(3, -1, -2)$ respectively.

Line and plane will be perpendicular iff $\frac{6}{3} = \frac{-2}{-1} = \frac{\lambda}{-2}$
$\Rightarrow \frac{6}{3} = \frac{\lambda}{-2} \Rightarrow \lambda = -4.$

**16.** Expanding along $R_1$, we get
$\Delta = 0 - \sin\alpha\,(0 - \sin\beta\cos\alpha) - \cos\alpha\,(\sin\alpha\sin\beta - 0)$
$= \sin\alpha\sin\beta\cos\alpha - \cos\alpha\sin\alpha\sin\beta = 0.$

**17.** Let $D_1, D_2$ be the events that we find a defective fuse in the first, second test respectively.
Required probability = $P(D_1 \cap D_2) = P(D_1)P(D_2 \mid D_1)$
$= \frac{2}{7}\cdot\frac{1}{6} = \frac{1}{21}$

**18.** $\int_0^{\pi/2} x\cos x\,dx = [x\sin x]_0^{\pi/2} - \int_0^{\pi/2} 1\cdot\sin x\,dx$
[Integrating by parts]
$= \frac{\pi}{2} + [\cos x]_0^{\pi/2} = \frac{\pi}{2} - 1$

**OR**

We have, $\int \cos^3 x\sin x\,dx$
Put $\cos x = t \Rightarrow \sin x\,dx = -dt$
$\therefore \ \int \cos^3 x\sin x\,dx = -\int t^3 dt = -\frac{t^4}{4} + C = -\frac{1}{4}\cos^4 x + C$

**19.** Let $y = \cos(\sin x)$
Now, $\frac{dy}{dx} = \frac{d\{\cos(\sin x)\}}{dx}$
$= -\sin(\sin x)\cdot\cos x = -\cos x\sin(\sin x)$

**20.** The given differential equation can be written as
$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3 = \left(\frac{5d^2y}{dx^2}\right)^2$
Clearly, order of differential equation is 2 and degree is 2.

## EXAM CORNER 2020

| Exam | Date |
|---|---|
| WB JEE | 2nd February |
| JEE Main II | 05th, 07th to 09th & 11th April |
| SRMJEEE | 12th to 20th April |
| VITEEE | 13th to 19th April |
| J&K CET | 19th April |
| Kerala PET | 20th & 21st April |
| Karnataka CET | 22nd & 23rd April |
| COMEDK (Engg.) | 10th May |
| AMU (Engg.) | 10th May |
| MHT CET | 16th to 20th May |
| BITSAT | 16th to 25th May |
| JEE Advanced | 17th May |

**21.** We have,

$$\tan^{-1}\left[\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right]=\tan^{-1}\left[\frac{\frac{a\cos x-b\sin x}{b\cos x}}{\frac{b\cos x+a\sin x}{b\cos x}}\right]$$

$$=\tan^{-1}\left[\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}\tan x}\right]=\tan^{-1}\frac{a}{b}-\tan^{-1}(\tan x)=\tan^{-1}\frac{a}{b}-x$$

**OR**

Given, $f(x) = x^2 + 5x + 9$. Let $f^{-1}(8) = x$. Then $f(x) = 8$

$\Rightarrow x^2 + 5x + 9 = 8 \Rightarrow x^2 + 5x + 1 = 0$

$\Rightarrow x = \dfrac{-5 \pm \sqrt{21}}{2} \therefore f^{-1}(8) = \left\{\dfrac{-5 \pm \sqrt{21}}{2}\right\}$

Again, let $f^{-1}(9) = x$

$\Rightarrow f(x) = 9 \Rightarrow x^2 + 5x + 9 = 9$

$\Rightarrow x^2 + 5x = 0 \Rightarrow x = 0, -5 \therefore f^{-1}(9) = \{-5, 0\}$.

**22.** Differentiating $x^{2/3} + y^{2/3} = 2$ with respect to $x$, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

$\therefore$ Slope of the tangent at $(1, 1) = -1$

Also, the slope of the normal at $(1, 1)$ is given by

$$\frac{-1}{\text{slope of the tangent at } (1, 1)} = 1$$

Therefore, the equation of the normal at $(1, 1)$ is

$y - 1 = 1(x - 1) \Rightarrow y - x = 0$

**23.** We have, $P$ (at least one of $A$ and $B$) $= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)\,P(B)$

$= P(A) + P(B)\,[1 - P(A)] = P(A) + P(B)\,P(A')$

$= 1 - P(A') + P(B)\,P(A') = 1 - P(A')\,[1 - P(B)]$

$= 1 - P(A')\,P(B')$

**24.** Here $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$

Since $\vec{a} \| \vec{b}$, therefore $\vec{a} = \lambda\vec{b}$

$\Rightarrow 3\hat{i} + 2\hat{j} + 9\hat{k} = \lambda(\hat{i} + p\hat{j} + 3\hat{k})$

$\Rightarrow \lambda = 3, 2 = \lambda p, 9 = 3\lambda \Rightarrow \lambda = 3, p = 2/3$

**OR**

Volume of parallelepiped $= [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$

$= 1(1 - 2) - 1(-1 - 1) + 1(2 + 1) = -1 + 2 + 3 = 4$

**25.** Any point on the line $\dfrac{x+2}{3} = \dfrac{y+1}{2} = \dfrac{z-3}{2} = \lambda$ (say)

is of the form $Q(3\lambda - 2, 2\lambda - 1, 2\lambda + 3)$ ...(i)

Now, distance $PQ$, where coordinates of $P$ are $(1, 3, 3)$, is 5.

So, $(3\lambda - 2 - 1)^2 + (2\lambda - 1 - 3)^2 + (2\lambda + 3 - 3)^2 = 5^2$

$\Rightarrow 9\lambda^2 + 9 - 18\lambda + 4\lambda^2 + 16 - 16\lambda + 4\lambda^2 = 25$

$\Rightarrow 17\lambda^2 - 34\lambda = 0 \Rightarrow 17\lambda(\lambda - 2) = 0$

$\Rightarrow \lambda = 0$ or $\lambda = 2$

Putting values of $\lambda$ in (i), the required points are $(-2, -1, 3)$ and $(4, 3, 7)$.

**26.** We have, R.H.L. (at $x = 2$)

$= \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2} (ax^2 + b) = 4a + b$

L.H.L. (at $x = 2$)

$= \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2} (2ax - b) = 4a - b$

and $f(2) = 2$

Since $f(x)$ is continuous at $x = 2$.

$\therefore \quad 4a + b = 2$ and $4a - b = 2$

Solving, we get $a = \dfrac{1}{2}$ and $b = 0$.

Thus, $f(x)$ is continuous at $x = 2$ if $a = \dfrac{1}{2}$ and $b = 0$.

**27.** Let $y = f(x) = 4x^2 + 12x + 15, x \in N$

Now, consider $y = (2x)^2 + 2 \times 2x \times 3 + 9 + 6$

$\Rightarrow y = (2x + 3)^2 + 6 \Rightarrow (2x + 3)^2 = y - 6$

$\Rightarrow 2x + 3 = \sqrt{y-6}$ [Taking positive square root]

$\Rightarrow x = \dfrac{\sqrt{y-6} - 3}{2}$ [$\sqrt{y-6}$ is defined for $y > 6$]

Let $g : S \to N$ be defined by

$$g(y) = \frac{\sqrt{y-6} - 3}{2} \quad \text{...(i)}$$

Now, $gof(x) = g(4x^2 + 12x + 15) = g[(2x + 3)^2 + 6]$

$$= \frac{\sqrt{(2x+3)^2 + 6 - 6} - 3}{2} \quad \text{[From (i)]}$$

$\Rightarrow gof(x) = I_N$ and $fog(y) = f\left(\dfrac{\sqrt{y-6} - 3}{2}\right)$

$$= \left[\frac{2\{\sqrt{(y-6)} - 3\}}{2} + 3\right]^2 + 6 = (\sqrt{y-6})^2 + 6 = y$$

$\Rightarrow fog(y) = I_S$

Hence, $f$ is invertible and inverse of $f$ *i.e.*,

$$f^{-1}(x) = \frac{\sqrt{x-6} - 3}{2}.$$

**28.** Continuity at $x = -3$ :

$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3} |x| + 3 = \lim_{x \to -3} (-x + 3) = 3 + 3 = 6$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3} (-2x) = 6$$

$f(-3) = |-3| + 3 = 3 + 3 = 6$

Thus, $\lim_{x\to -3^-} f(x) = \lim_{x\to -3^+} f(x) = f(-3)$

$\therefore$ $f$ is continuous at $x = -3$.

Continuity at $x = 3$ :

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3} (-2x) = -6$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3} (6x+2) = 6(3)+2 = 20$

Thus, $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$

$\therefore$ $f(x)$ is discontinuous at $x = 3$.

So, the only point of discontinuity of $f$ is $x = 3$.

**OR**

*Refer to answer 27, Page no. 104 of MTG CBSE Champion Mathematics, Class-12.*

**29.** Let $I = \int \frac{x^2+x+1}{(x^2+1)(x+2)}dx$ ...(i)

Let $\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2}$ ...(ii)

$\Rightarrow$ $x^2 + x + 1 = (Ax + B)(x + 2) + C(x^2 + 1)$

Put $x = 0, 1, -2$ in it to get

$1 = 2B + C$ ; $3 = 3(A + B) + 2C$ and $3 = 5C$

$\Rightarrow$ $C = \frac{3}{5}, B = \frac{1}{5}$ and $A = \frac{2}{5}$

Hence, from (ii), we have

$$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{\left(\frac{2}{5}x+\frac{1}{5}\right)}{x^2+1} + \frac{\frac{3}{5}}{x+2} = \frac{1}{5}\cdot\frac{2x+1}{x^2+1} + \frac{3}{5}\cdot\frac{1}{x+2}$$

$\therefore$ $I = \frac{1}{5}\int \frac{2x+1}{x^2+1}dx + \frac{3}{5}\int \frac{1}{x+2}dx$

$= \frac{1}{5}\left[\int \frac{2x}{x^2+1}dx + \int \frac{dx}{x^2+1}\right] + \frac{3}{5}\int \frac{dx}{x+2}$

$= \frac{1}{5}\left[\log|x^2+1| + \tan^{-1}x\right] + \frac{3}{5}\log|x+2| + C_1$

**30.** We have, $y^2\,dx + (x^2 - xy + y^2)\,dy = 0$

$\Rightarrow$ $\frac{dy}{dx} = \frac{-y^2}{x^2-xy+y^2}$ ...(i)

This is homogeneous differential equation.

$\therefore$ Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

From (i), $v + x\frac{dv}{dx} = \frac{-v^2x^2}{x^2 - vx^2 + v^2x^2}$

$\Rightarrow$ $v + x\frac{dv}{dx} = \frac{-v^2}{1-v+v^2}$ $\Rightarrow$ $x\frac{dv}{dx} = \frac{-v^2}{1-v+v^2} - v$

$\Rightarrow$ $x\frac{dv}{dx} = \frac{-v-v^3}{1-v+v^2}$ $\Rightarrow$ $\frac{1-v+v^2}{v(1+v^2)}dv = -\frac{1}{x}dx$

Integrating both sides, we get

$$\int \frac{1+v^2}{v(1+v^2)}dv - \int \frac{v}{v(1+v^2)}dv = -\int \frac{1}{x}dx$$

$\Rightarrow$ $\int \frac{1}{v}dv - \int \frac{1}{1+v^2}dv = -\int \frac{1}{x}dx$

$\Rightarrow$ $\log|v| - \tan^{-1}v = -\log|x| + \log C$

$\Rightarrow$ $\log\left|\frac{vx}{C}\right| = \tan^{-1}v$ $\Rightarrow$ $\left|\frac{vx}{C}\right| = e^{\tan^{-1}v}$

$\Rightarrow$ $|y| = Ce^{\tan^{-1}(y/x)}$ is the required solution.

**31.** Total number of cards = 52

Number of ace cards = 4

Number of other cards = 52 – 4 = 48

Let $X$ be the number of ace cards drawn.

$\therefore$ $X$ can take the values 0, 1 or 2

$\therefore$ $P(X = 0) = P$ (no ace card)

$= \frac{{}^4C_0 \times {}^{48}C_2}{{}^{52}C_2} = \frac{48\times 47}{52\times 51} = \frac{188}{221}$

$P(X = 1) = P$ (one ace card)

$= \frac{{}^4C_1 \times {}^{48}C_1}{{}^{52}C_2} = \frac{4\times 48\times 2}{52\times 51} = \frac{32}{221}$

$P(X = 2) = P$ (two ace cards)

$= \frac{{}^4C_2 \times {}^{48}C_0}{{}^{52}C_2} = \frac{4\times 3}{52\times 51} = \frac{1}{221}$

Required probability distribution is

| $X$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(X)$ | $\frac{188}{221}$ | $\frac{32}{221}$ | $\frac{1}{221}$ |

Now, Mean = $\sum XP(X)$

$= 0\times\frac{188}{221} + 1\times\frac{32}{221} + 2\times\frac{1}{221} = \frac{34}{221} = \frac{2}{13}$

**OR**

*Refer to answer 27, Page no. 333 of MTG CBSE Champion Mathematics, Class-12.*

**Monthly Test Drive-9 CLASS XII** ANSWER KEY

| | | | |
|---|---|---|---|
| **1.** (c) | **2.** (d) | **3.** (b) | **4.** (a) |
| **5.** (b) | **6.** (c) | **7.** (b, c) | **8.** (b, d) |
| **9.** (b, c) | **10.** (b, c) | **11.** (a, b, c) | **12.** (a, d) |
| **13.** (b) | **14.** (c) | **15.** (d) | **16.** (d) |
| **17.** (0.25) | **18.** (3) | **19.** (–0.5) | **20.** (15) |

**32.** Let the dietician mix $x$ kg of food I and $y$ kg of food II.

Since 1 kg of food I costs ₹ 5 and 1 kg of food II costs ₹ 7.

$\therefore$ Total cost of $x$ kg of food I and $y$ kg of food II is ₹ $(5x + 7y)$.

$\therefore$ Minimize $z = 5x + 7y$

The given data may be put in the following tabular form:

| Vitamin | Food | | Requirement |
|---|---|---|---|
| | I | II | |
| $A$ | 2 | 1 | 8 |
| $C$ | 1 | 2 | 10 |
| Cost (₹) | 5 | 7 | |

Since, the minimum requirement of vitamin $A$ is 8 units.

$\therefore \quad 2x + y \geq 8$ ...(i)

Since, the minimum requirement of vitamin $C$ is 10 units.

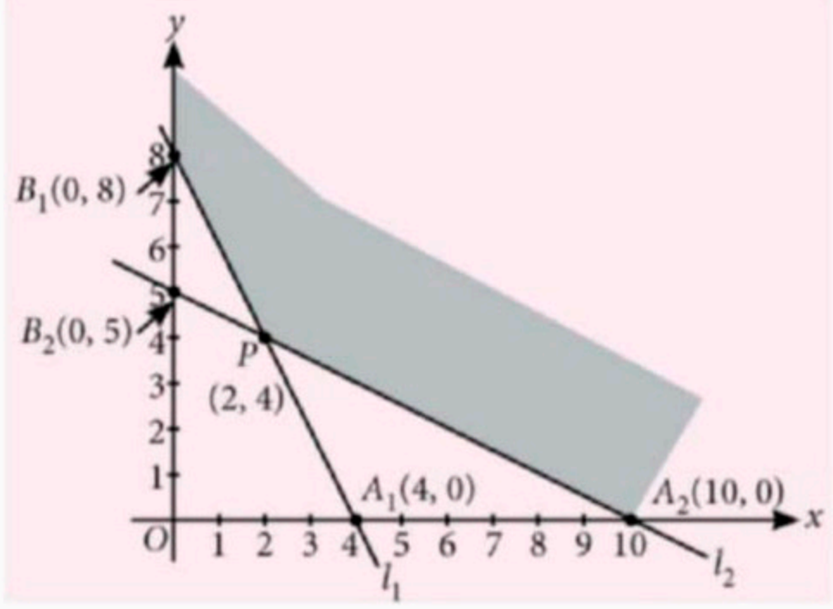
$\therefore \quad x + 2y \geq 10$ ...(ii)

Hence, the mathematical formation of the LPP is as follows:

Minimize $z = 5x + 7y$

subject to $2x + y \geq 8$, $x + 2y \geq 10$ and $x, y \geq 0$

To solve this LPP graphically, we first convert the inequations into equations to obtain the following lines $l_1 : 2x + y = 8$, $l_2 : x + 2y = 10$, $l_3 : x = 0$, $l_4 : y = 0$




The coordinates of the corner points of the feasible region are $A_2(10, 0)$, $P(2, 4)$, $B_1(0, 8)$.

The values of the objective function $z = 5x + 7y$ at the corner points of the feasible region are given in the following table.

| Corner Points | Value of $z = 5x + 7y$ |
|---|---|
| $A_2(10, 0)$ | 50 |
| $P(2, 4)$ | 38 (minimum) |
| $B_1(0, 8)$ | 56 |

Minimum cost = ₹ 38 at $x = 2$, $y = 4$

**33.** We have, $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

$= (a + b + c)\,[(c - b)\,(b - c) - (a - b)\,(a - c)]$

$= (a + b + c)\,[(bc - c^2 - b^2 + bc) - (a^2 - ac - ab + bc)]$

$= (a + b + c)\,[2bc - c^2 - b^2 - a^2 + ac + ab - bc]$

$= (a + b + c)\,[-a^2 - b^2 - c^2 + ab + bc + ca]$

$= -(a + b + c)\,[a^2 + b^2 + c^2 - ab - bc - ca]$

$= -(a+b+c)\cdot\frac{1}{2}\cdot 2(a^2+b^2+c^2-ab-bc-ca)$

$= -\frac{1}{2}(a+b+c)\cdot[(a-b)^2+(b-c)^2+(c-a)^2]$, which is clearly negative as $a,b,c$ are given to be positive and unequal.

**OR**

*Refer to answer 124, Page no. 85 of MTG CBSE Champion Mathematics, Class-12.*

**34.** The given circles are

$C_1 : x^2 + y^2 = 4$ ...(i) and $C_2 : (x - 2)^2 + y^2 = 4$ ...(ii)

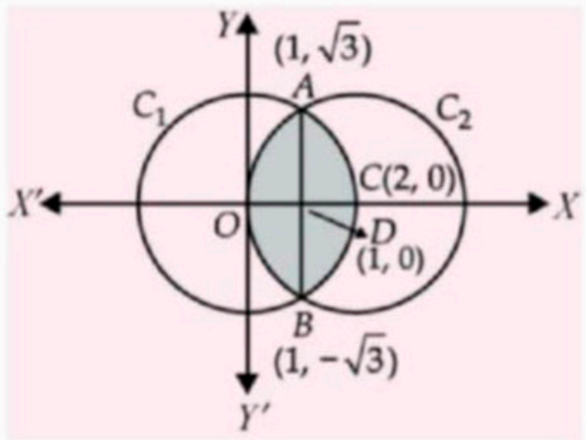
Eliminating $y$ from (i) and (ii), we get

$4 - x^2 = 4 - (x - 2)^2$

$\Rightarrow \quad 4x = 4 \Rightarrow x = 1$

Putting $x = 1$ in (i), we get $y^2 = 3$

$\Rightarrow \quad y = \pm\sqrt{3}$




$\therefore$ Points of intersection of the two circles are $A(1, \sqrt{3})$ and $B(1, -\sqrt{3})$.

Required area = 2(area $AOCA$)

= 2(area $AODA$ + area $ADCA$)

$$= 2\int_0^1 \sqrt{4-(x-2)^2}\,dx + 2\int_1^2 \sqrt{4-x^2}\,dx$$

$$= 2\left[\frac{x-2}{2}\sqrt{4-(x-2)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$$

$$+2\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_1^2$$

$$= 2\left[-\frac{\sqrt{3}}{2} + 2\left(\frac{-\pi}{6}\right) - 2\left(-\frac{\pi}{2}\right) + 2\cdot\frac{\pi}{2} - \frac{\sqrt{3}}{2} - \frac{2\times\pi}{6}\right]$$

$$= 2\left[-\sqrt{3} + 2\pi - \frac{2\pi}{3}\right] = 2\left[\frac{-3\sqrt{3}+4\pi}{3}\right] \text{ sq. units}$$

**35.** We have, $f(x) = \sin x - \cos x, 0 \le x \le 2\pi$

For critical points, $f'(x) = 0$

$\therefore \quad \cos x + \sin x = 0 \Rightarrow \sin x = -\cos x \Rightarrow \tan x = -1$

$\therefore \quad \tan x = \tan\left(-\frac{\pi}{4}\right) \Rightarrow x = n\pi - \frac{\pi}{4}$

As $x \in [0, 2\pi] \quad \therefore \; x = \frac{3\pi}{4}, \frac{7\pi}{4}$

The point $x = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$ divide the interval $[0, 2\pi]$ into the following disjoint intervals

$$\left[0, \frac{3\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{7\pi}{4}\right) \text{ and } \left(\frac{7\pi}{4}, 2\pi\right]$$

| Interval | Sign of $f'(x)$ | Nature of function |
|---|---|---|
| $\left[0, \frac{3\pi}{4}\right)$ | $> 0$ | Strictly increasing |
| $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$ | $< 0$ | Strictly decreasing |
| $\left(\frac{7\pi}{4}, 2\pi\right]$ | $> 0$ | Strictly increasing |

Hence, $f(x)$ is strictly increasing on $\left[0, \frac{3\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right]$ and $f(x)$ is strictly decreasing on $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$.

**OR**

*Refer to answer 91, Page no. 147 of MTG CBSE Champion Mathematics, Class-12.*

**36.** Given planes are $2x + 3y - 2z = 5$ ...(i)

and $x + 2y - 3z = 8$ ...(ii)

Normal vectors of (i) and (ii) are respectively

$\vec{m}_1 = 2\hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{m}_2 = \hat{i} + 2\hat{j} - 3\hat{k}$

Since, required plane is perpendicular to the planes (i) and (ii). So, normal to the required plane will be in the direction of $\vec{m}_1 \times \vec{m}_2$.

So, $\vec{m}_1 \times \vec{m}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -2 \\ 1 & 2 & -3 \end{vmatrix} = -5\hat{i} + 4\hat{j} + \hat{k} = \vec{m}$ (say)

Also, position vector of point (1, –1, 2) on the plane is, $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$.

So, equation of plane is $\vec{r} \cdot \vec{m} = \vec{a} \cdot \vec{m}$

$\Rightarrow \quad \vec{r} \cdot (-5\hat{i} + 4\hat{j} + \hat{k}) = (\hat{i} - \hat{j} + 2\hat{k}) \cdot (-5\hat{i} + 4\hat{j} + \hat{k})$

$\Rightarrow \quad 5x - 4y - z - 7 = 0$ ...(iii)

Distance of $P(-2, 5, 5)$ from (iii) is,

$$d = \left|\frac{5(-2) - 4(5) - 5 - 7}{\sqrt{5^2 + (-4)^2 + (-1)^2}}\right| = \frac{42}{\sqrt{42}} = \sqrt{42} \text{ units.}$$

## CBSE 2020 Board Examination :

### Preparation tips and strategies for Class 10, 12 students

The Central Board of Secondary Education (CBSE) on December 18, Tuesday, announced the CBSE Class 10 and Class 12 examination schedule. The CBSE 10th 2020 examination schedule, CBSE 12th 2020 examination schedule were released by the exam conducting authority on the official website. The CBSE 10th and 12th examination for the year 2019-20 will begin from February 15.

- Prepare a timetable for yourself. While preparing it, make sure you have included Mathematics and English in your daily curriculum.
- Give adequate time to each subject and don't forget to take a 10-minute break every 2 hours. Try not to study too much at a stretch.
- If there is a subject that needs extra care, repeat it on your timetable 4 times a week.
- For each subject, prepare a list of topics that need to be worked on. Then divide the topics into two parts - like and dislike. Don't forget to tick off the topic after you complete them.
- Use different techniques for revision. Try to break your notes to keywords and phrases and then revise.
- Set a daily goal for yourself and try studying at the same time daily.
- Students can download the sample question papers released by the board on its official website and solve it.
- Refer to model question papers for each subject and prepare accordingly.
- Solve the last five years' board question papers and to get yourself familiar with the pattern. Always remember to sit with a stop watch while solving question papers.
- Organise group study time to exchange ideas and thoughts and be successful in your exams.

*Courtesy : Hindustan Times*

# MONTHLY TEST DRIVE

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

**Total Marks : 80**

## Series 9 : Conic Sections

**Time Taken : 60 Min.**

### Only One Option Correct Type

1. If the distances from the origin of the centres of three circles $x^2 + y^2 + 2\lambda_i x - c^2 = 0$ $(i = 1, 2, 3)$ are in G.P., then the lengths of the tangents drawn to them from any point on the circle $x^2 + y^2 = c^2$ are in
(a) A.P. (b) G.P.
(c) H.P. (d) None of these

2. The length of the latusrectum of the parabola $169\{(x-1)^2 + (y-3)^2\} = (5x - 12y + 7)^2$, is
(a) $\frac{14}{13}$ (b) $\frac{28}{13}$ (c) $\frac{12}{13}$ (d) $\frac{48}{13}$

3. If the tangent from a point $P$ to the circle $x^2 + y^2 = 1$ is perpendicular to the tangent from $P$ to the circle $x^2 + y^2 = 3$, then the locus of $P$, is
(a) a circle of radius 2 (b) a circle of radius 4
(c) a circle of radius 3 (d) none of these

4. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and having centre at $(0, 3)$, is
(a) $x^2 + y^2 - 6y - 7 = 0$ (b) $x^2 + y^2 - 6y + 7 = 0$
(c) $x^2 + y^2 - 6y - 5 = 0$ (d) $x^2 + y^2 - 6y + 5 = 0$

5. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the $x$-axis, then the eccentricity of the hyperbola is
(a) $\sqrt{2}$ (b) 2 (c) $\sqrt{3}$ (d) 1

6. If $PSQ$ is a focal chord of the ellipse $16x^2 + 25y^2 = 400$ such that $SP = 8$, then $SQ =$
(a) 1 (b) 2 (c) 3 (d) 4

### One or More Than One Option(s) Correct Type

7. If the area of the quadrilateral formed by the tangents from the origin to the circle $x^2 + y^2 + 6x - 10y + c = 0$ and the pair of radii at the points of contact of these tangents to the circle is 8 sq. units, then the value of $c$ is
(a) 2 (b) 4 (c) 16 (d) 32

8. Let $E_1$ and $E_2$ be two ellipses whose centres are at the origin. The major axes of $E_1$ and $E_2$ lie along the $x$-axis and the $y$-axis, respectively. Let $S$ be the circle $x^2 + (y-1)^2 = 2$. The straight line $x + y = 3$ touches the curves $S$, $E_1$ and $E_2$ at $P$, $Q$ and $R$, respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If $e_1$ and $e_2$ are the eccentricities of $E_1$ and $E_2$, respectively, then the correct expression(s) is (are)
(a) $e_1^2 + e_2^2 = \frac{43}{40}$ (b) $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$
(c) $\left|e_1^2 - e_2^2\right| = \frac{5}{8}$ (d) $e_1 e_2 = \frac{\sqrt{3}}{4}$

9. If $(5, 12)$ and $(24, 7)$ are the foci of a hyperbola passing through the origin, then
(a) $e = \sqrt{386}/12$ (b) $e = \sqrt{386}/13$
(c) LR = 121/6 (d) LR = 121/3

10. The parabola $y^2 = 4x$ and the circle having its centre at $(6, 5)$ intersect at right angle. The possible point of intersection of these curves can be
(a) $(9, 6)$ (b) $(2, \sqrt{8})$ (c) $(4, 4)$ (d) $(3, 2\sqrt{3})$

11. The equations of four circles are $(x \pm a)^2 + (y \pm a)^2 = a^2$. The radius of a circle touching all the four circles is

(a) $(\sqrt{2}-1)a$ (b) $2\sqrt{2}a$

(c) $(\sqrt{2}+1)a$ (d) $(2+\sqrt{2})a$

**12.** Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ parallel to the straight line $2x-y=1$. The points of contact of the tangents on the hyperbola are

(a) $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (b) $\left(-\frac{9}{2\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$

(c) $\left(3\sqrt{3},-2\sqrt{2}\right)$ (d) $\left(3\sqrt{3},2\sqrt{2}\right)$

**13.** Let $A$ and $B$ be two distinct points on the parabola $y^2=4x$. If the axis of the parabola touches a circle of radius $r$ having $AB$ as its diameter, then the slope of the line joining $A$ and $B$ can be

(a) $-1/r$ (b) $1/r$ (c) $2/r$ (d) $-2/r$

## Comprehension Type

$P$ is a variable point on the line $L=0$. Tangents are drawn to the circle $x^2+y^2=4$ from $P$ to touch it at $Q$ and $R$. The parallelogram $PQSR$ is completed.

**14.** If $L\equiv 2x+y=6$, then the locus of circumcentre of $\Delta PQR$ is

(a) $2x-y=4$ (b) $2x+y=3$

(c) $x-2y=4$ (d) $x+2y=3$

**15.** If $P\equiv(6,8)$, then the area of $\Delta QRS$ is

(a) $\frac{(6)^{3/2}}{25}$ sq. unit (b) $\frac{(24)^{3/2}}{25}$ sq. unit

(c) $\frac{48\sqrt{6}}{25}$ sq. units (d) $\frac{192\sqrt{6}}{25}$ sq. units

## Matrix Match Type

**16.** Match the following columns:

| Column-I | | Column-II | |
|---|---|---|---|
| P. | If the circumference of the circle $x^2+y^2-2x+8y-\lambda=0$ is bisected by the circle $x^2+y^2+4x+22y+\lambda=0$, then $\lambda$ equals | 1. | –12 |
| Q. | If the chord $x+2y+\lambda=0$ subtends a right angle at the vertex of the parabola $x^2=6y$, then $\lambda$ equals | 2. | 24 |
| R. | A tangent having a slope $-\frac{4}{3}$ to the ellipse $\frac{x^2}{18}+\frac{y^2}{32}=1$ intersects the major and minor axes at $A$ and $B$. If $O$ is the origin, then the area (in sq. units) of $\Delta OAB$ is | 3. | 16 |
| S. | If a circle cuts a rectangular hyperbola $xy=c^2$ in $A$, $B$, $C$ and $D$ and the parameters of these four points be $t_1$, $t_2$, $t_3$ and $t_4$ respectively, then the value of $16t_1t_2t_3t_4$ must be | 4. | 25 |

| | P | Q | R | S |
|---|---|---|---|---|
| (a) | 2 | 4 | 1 | 3 |
| (b) | 1 | 3 | 4 | 2 |
| (c) | 4 | 1 | 3 | 2 |
| (d) | 4 | 1 | 2 | 3 |

## Numerical Value Type

**17.** The line passing through the extremity $A$ of the major axis and extremity $B$ of the minor axis of the ellipse $x^2+9y^2=9$ meets the auxiliary circle at the point $M$. Then the area (in sq. units) of the triangle with vertices at $A$, $M$ and the origin $O$, is

**18.** Let $a$ and $b$ respectively be the semi-transverse and semi-conjugate axes of a hyperbola whose eccentricity $e$ satisfies the equation $9e^2-18e+5=0$. If $S(5,0)$ is a focus and $5x=9$ is the corresponding directrix of hyperbola, then $a^2-b^2$ is equal to

**19.** If $\left(m_i,\frac{1}{m_i}\right)$, $m_i>0, i=1,2,3,4$ are four distinct points on a circle, then the value of $\prod_{i=1}^{4} m_i$ must be

**20.** The normal to the parabola $y^2=8x$ at the point $(2,4)$ meets it again at $(\alpha,\beta)$. If length of normal chord is $\lambda$, then the value of $\lambda^2$ must be

*Keys are published in this issue. Search now!* ☺

**SELF CHECK**

No. of questions attempted .....
No. of questions correct .....
Marks scored in percentage .....

Check your score! If your score is

| | | |
|---|---|---|
| > 90% | EXCELLENT WORK ! | You are well prepared to take the challenge of final exam. |
| 90-75% | GOOD WORK ! | You can score good in the final exam. |
| 74-60% | SATISFACTORY ! | You need to score more next time. |
| < 60% | NOT SATISFACTORY! | Revise thoroughly and strengthen your concepts. |

# YOU ASK WE ANSWER

**Do you have a question that you just can't get answered?**

Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough. The best questions and their solutions will be printed in this column each month.

**1.** Evaluate : $\int_0^{\sin^2 x} \sin^{-1}(\sqrt{t})dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t})dt$.

*(Anand Mishra, Delhi)*

**Ans.** Let $I = \int_0^{\sin^2 x} \sin^{-1}(\sqrt{t})dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t})dt$

$$= [t\sin^{-1}(\sqrt{t})]_0^{\sin^2 x} - \int_0^{\sin^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}dt + [t\cos^{-1}(\sqrt{t})]_0^{\cos^2 x} + \int_0^{\cos^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}dt$$

$$= x\sin^2 x + \int_{\sin^2 x}^{0} \frac{\sqrt{t}}{2\sqrt{1-t}}dt + x\cos^2 x + \int_0^{\cos^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}dt$$

$$= x(\sin^2 x + \cos^2 x) + \int_{\sin^2 x}^{\cos^2 x} \frac{\sqrt{t}}{2\sqrt{1-t}}dt$$

Putting $t = \sin^2\theta$ and $dt = 2\sin\theta\cos\theta\, d\theta$, we have

$$\int \frac{\sqrt{t}}{2\sqrt{1-t}}dt = \int \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\sin\theta\cos\theta\, d\theta$$

$$= \int \sin^2\theta\, d\theta = \int \frac{1-\cos 2\theta}{2}d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}$$

Also, when $t = \sin^2 x$, then $\theta = x$ and when $t = \cos^2 x$, then $\theta = \frac{\pi}{2} - x$.

Hence we have $I = x + \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_x^{(\pi/2)-x}$

$$= x + \left(\frac{\pi}{4} - \frac{x}{2} - \frac{\sin 2x}{4}\right) - \left(\frac{x}{2} - \frac{\sin 2x}{4}\right) = x + \frac{\pi}{4} - x = \frac{\pi}{4}$$

**2.** If $f(x) = x^3 + (a-3)x^2 + \frac{4}{3}(a-4)x + 5$ has a maxima at some $x \in R^-$ and minima at some $x \in R^+$, then find the possible values of $a$.

*(Sakshi Verma, Jharkhand)*

**Ans.** We have, $f(x) = x^3 + (a-3)x^2 + \frac{4}{3}(a-4)x + 5$

and $f'(x) = 3x^2 + 2(a-3)x + \frac{4}{3}(a-4)$

According to the given condition, $f'(x)$ must vanish at two distinct points say $\alpha$, $\beta$ such that $\alpha < 0$ and $\beta > 0$. Then, we have

$$f'(x) = 3(x-\alpha)(x-\beta)$$

and $f''(x) = 3(2x - \alpha - \beta)$

According to the given condition, $\alpha$ is a point of maxima

$\therefore \quad f''(\alpha) < 0 \Rightarrow 3(\alpha - \beta) < 0$, which is true $\forall\, a$

Also, $\beta$ is a point of minima

$\therefore \quad f''(\beta) > 0 \Rightarrow 3(\beta - \alpha) > 0$, which is true $\forall\, a$

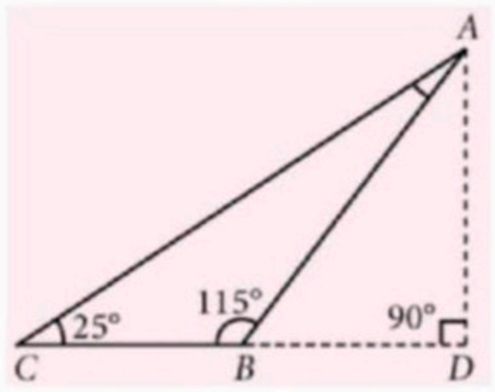
Now, for $\alpha < 0$ and $\beta > 0$, we have

$$\alpha\beta < 0 \Rightarrow \frac{4}{9}(a-4) < 0$$

$\Rightarrow \quad a < 4$ or $a \in (-\infty, 4)$.

**3.** If the base $BC$ of a $\Delta ABC$ is 6 cm, $\angle B = 115°$ and $\angle C = 25°$, then find its altitude. *(Pratiksha, W.B.)*

**Ans.** Extend $BC$ upto $D$, such that $AD$ is perpendicular to $BD$.




In $\Delta ABC$, $\angle CAB = 180° - (25° + 115°) = 40°$

In $\Delta ADC$, $\frac{AD}{AC} = \sin C$

$\Rightarrow \quad AD = AC\sin C \quad ...(i)$

In $\Delta ABC$, using sine rule

$$\frac{AB}{\sin C} = \frac{BC}{\sin A} = \frac{AC}{\sin B} \Rightarrow AC = BC\frac{\sin B}{\sin A}$$

From (i), $AD = BC\frac{\sin B \sin C}{\sin A} = BC\frac{(2\sin 115°\sin 25°)}{2\sin 40°}$

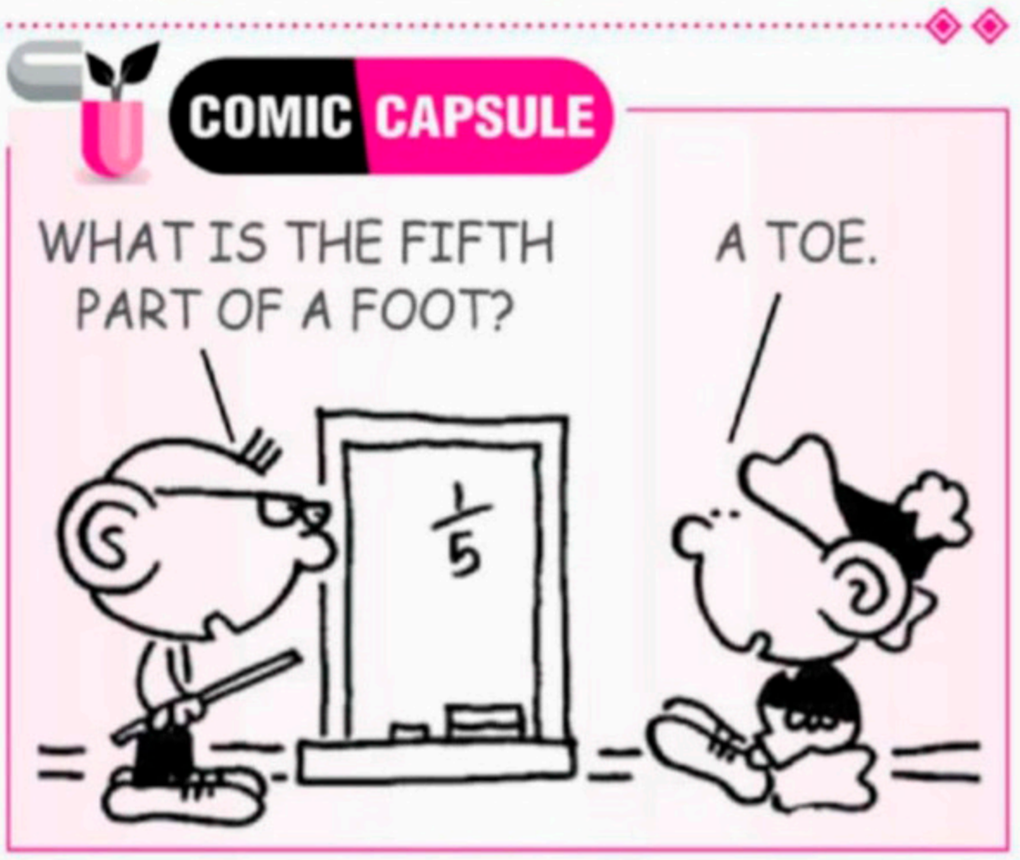
$$= BC\frac{(\cos 90° - \cos 140°)}{2\sin 40°} = BC\frac{\{0 - \cos(180° - 40°)\}}{2\sin 40°}$$

$$= \frac{6\cos 40°}{2\sin 40°} \qquad [\because BC = 6 \text{ cm (Given)}]$$

$= 3\cot 40° = 3 \times 1.19 \qquad [\because \cot 40° = 1.19]$

$= 3.57$ cm

## COMIC CAPSULE



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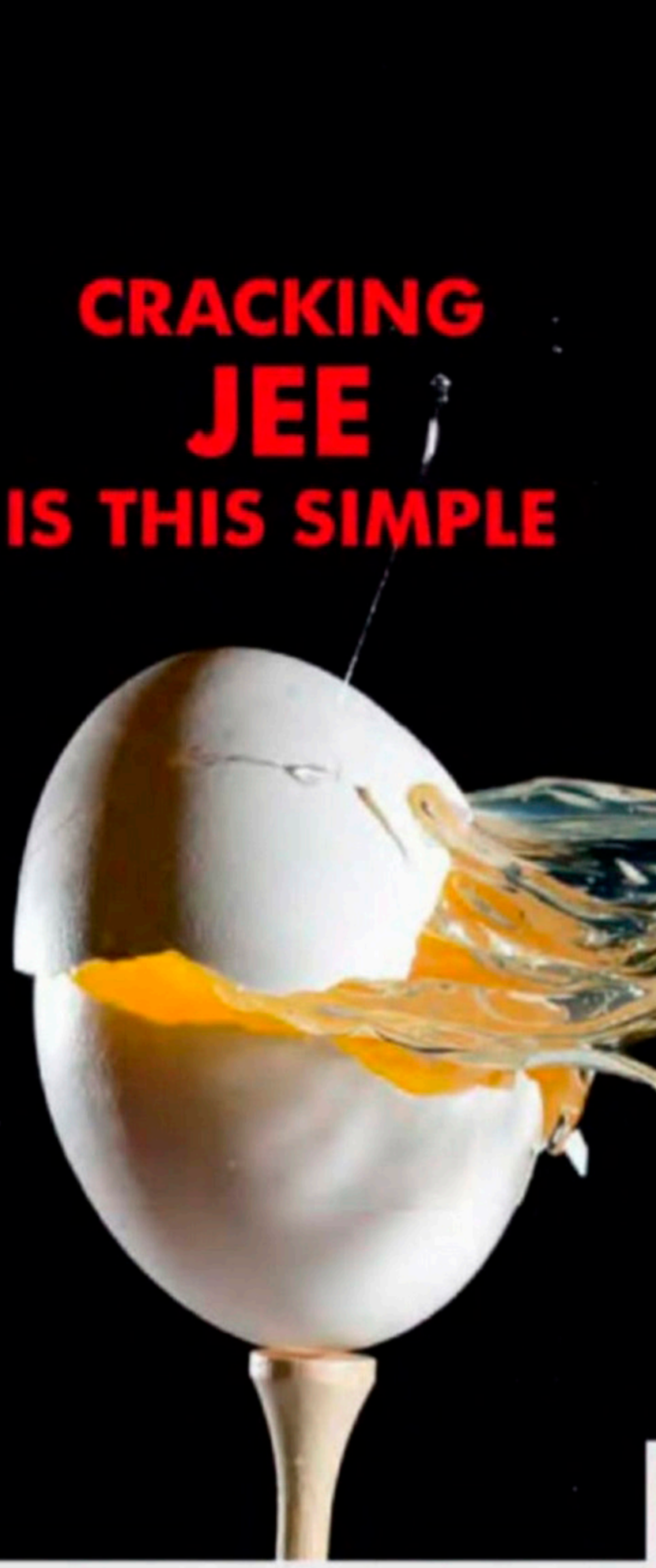

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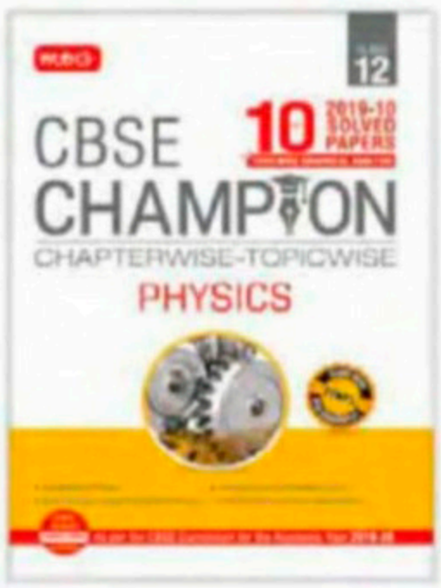


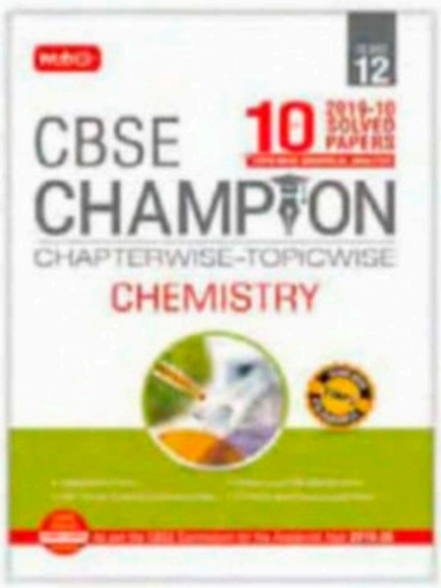


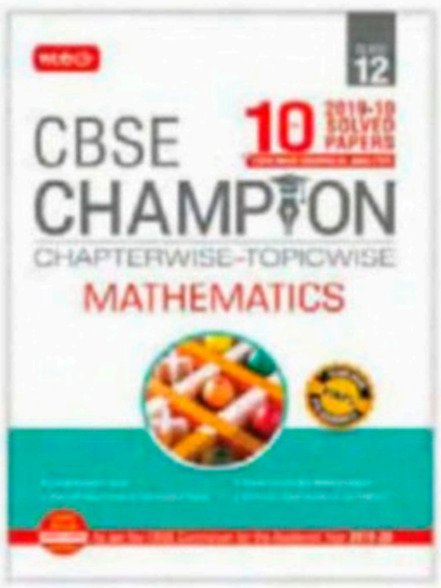


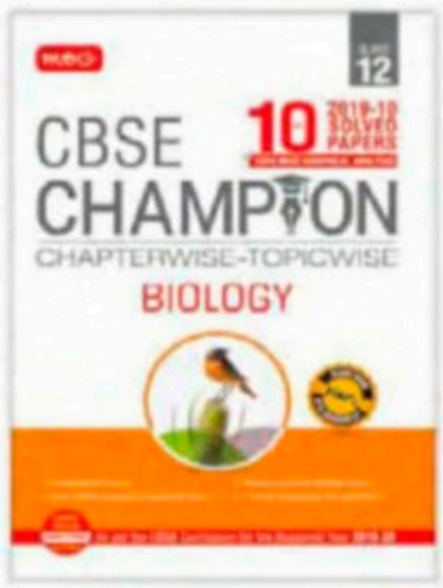


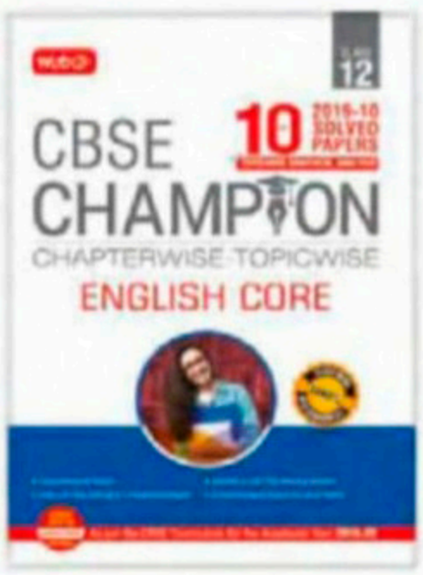


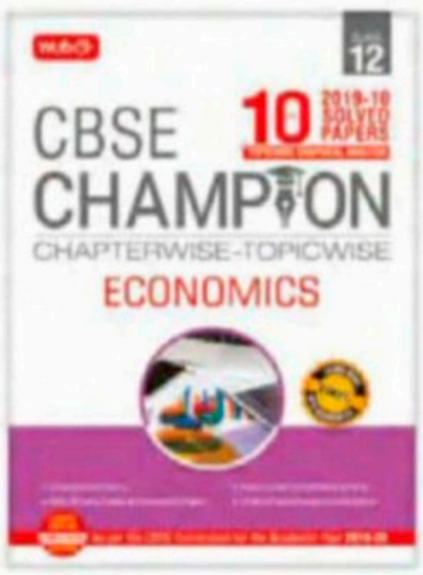


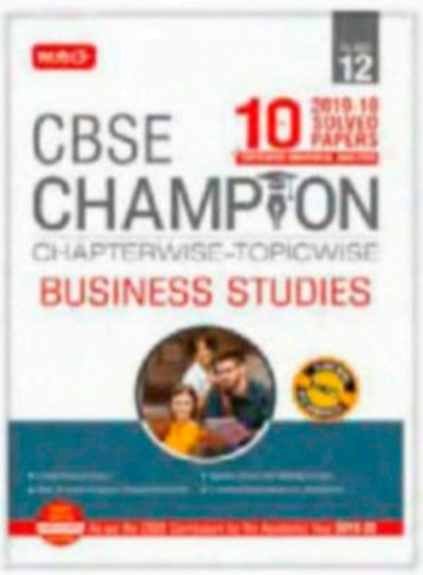


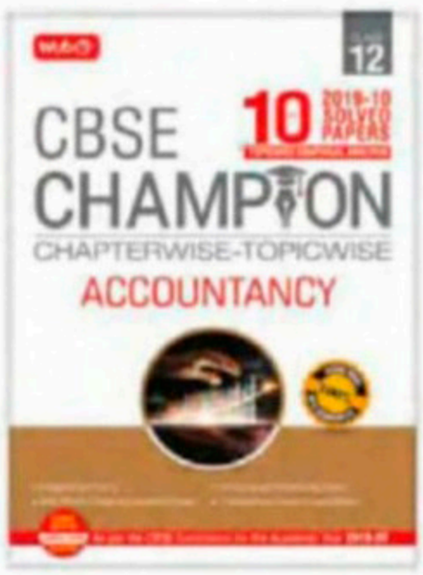


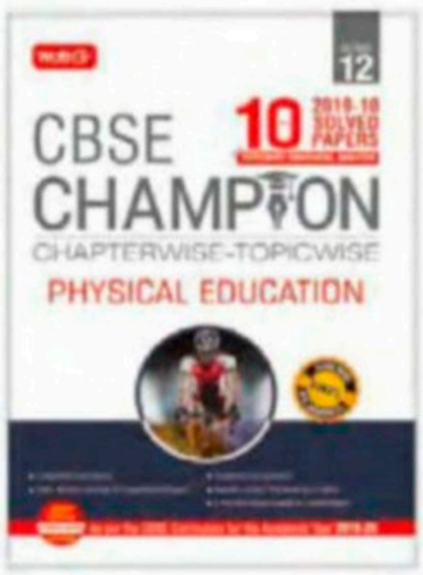

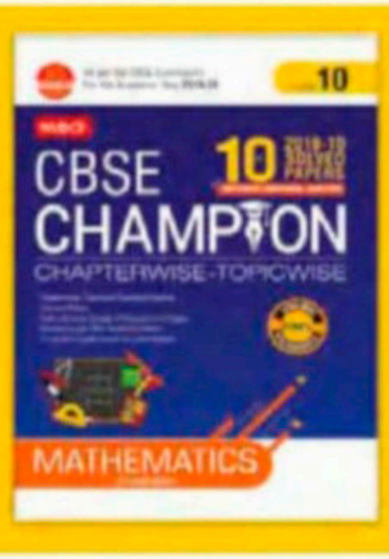

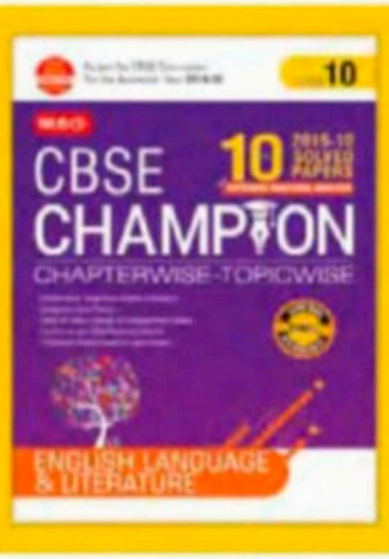

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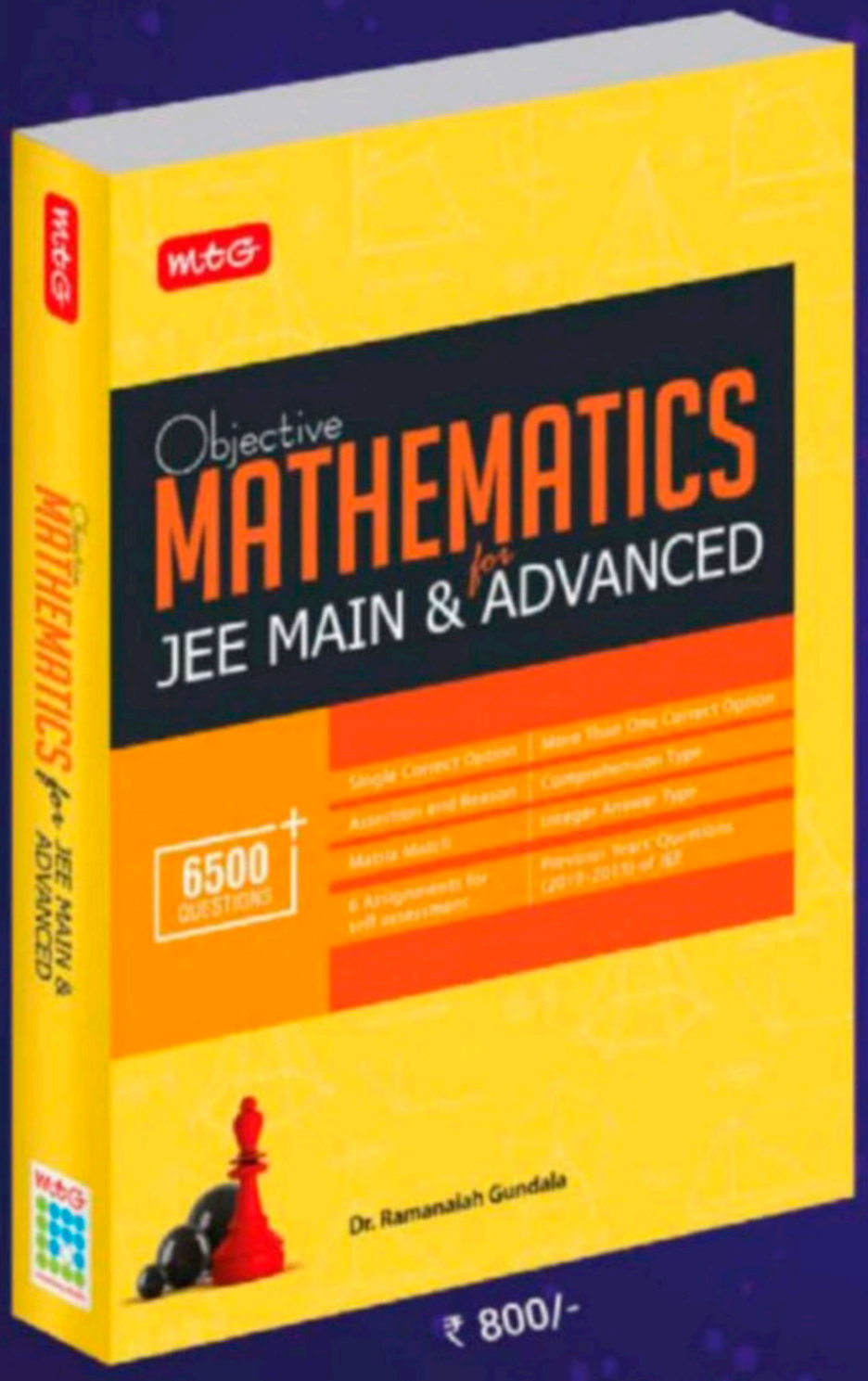


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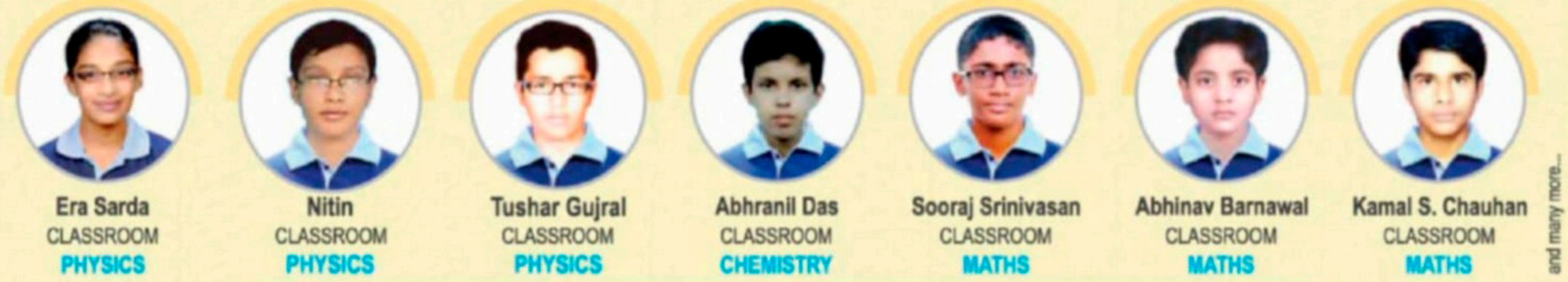

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